Vector properties.

If $\nabla \times \vec{F} = \vec{0}$, can replace $\vec{F}$ by $\vec{\nabla}\phi$.

If $\nabla \cdot \vec{F} = 0$, can replace $\vec{F}$ by $\vec{\nabla} \times \vec{A}$ &

can impose one condition on $\vec{A}$.

(Because you add any grad $\vec{\nabla}\psi$ to $\vec{A}$, & $\vec{F}' = \nabla \times \vec{A}'$.)

"Gauge" transformation.

Maxwell's Equations

$$\vec{\nabla} \cdot \vec{D} = \rho \qquad \vec{D} = \epsilon \vec{E}$$

ρ charge density

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \times \vec{H} = \vec{j} + \frac{\partial \vec{D}}{\partial t}$$

$\vec{j}$ = current flux $= \rho \vec{v}$

$$\vec{H} = \vec{B}/\mu$$

Conservation charge

$$\frac{\partial \rho}{\partial t} = -\vec{\nabla} \cdot \vec{j} = i\omega\rho$$

Replace $\vec{D}, \vec{H}$ by $\epsilon\vec{E}, \vec{B}/\mu$

$$\vec{\nabla}\cdot\vec{E} = \rho/\epsilon$$

In frequency domain: all $\propto e^{i\omega t}$

$$\vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} = -i\omega\vec{B}$$

$$\vec{\nabla}\times\vec{B} = \mu\vec{j} + \mu\epsilon\frac{\partial\vec{E}}{\partial t} = \mu\vec{j} + i\omega\mu\epsilon\vec{E}$$

Redundancy in M's eqs.

take div of $\vec{\nabla} \times \vec{H} = \vec{j} + i\omega \vec{D}$

$$\nabla \cdot \text{curl} = 0 = \underbrace{\vec{\nabla} \cdot \vec{j}}_{= -i\omega\rho} + i\omega \vec{\nabla} \cdot \vec{D}$$

$i\omega\rho = i\omega \vec{\nabla} \cdot \vec{D}$, so $\vec{\nabla} \cdot \vec{D} = \rho$ follows, if $\omega \neq 0$.

So 1st eq is redundant for AC, but not for static DC.

2nd eq.

$\vec{\nabla}\cdot\vec{B} = 0$ so $\vec{B}$ can be replaced by $\vec{\nabla}\times\vec{A}$.

3rd eq.

$$\vec{\nabla}\times\vec{E} = -i\omega\vec{B} = -i\omega\vec{\nabla}\times\vec{A}$$

$$\vec{\nabla}\times\left[\vec{E} + i\omega\vec{A}\right] = 0$$

$$\Longrightarrow \vec{E} + i\omega\vec{A} = -\nabla\phi$$

$$\vec{\nabla}\times\vec{B} = \mu\vec{j} + i\omega\epsilon\mu\vec{E}$$

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \mu\vec{j} + i\omega\epsilon\mu\left[-i\omega\vec{A} - \nabla\phi\right]$$

$$\vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A} = \mu\vec{j} + \omega^2\epsilon\mu\vec{A} - i\omega\epsilon\mu\vec{\nabla}\phi$$

$$\vec{\nabla}\underbrace{\left[\vec{\nabla}\cdot\vec{A} + i\omega\epsilon\mu\phi\right]}_{\text{make this zero:}} - \left\{\nabla^2\vec{A} + \omega^2\epsilon\mu\vec{A}\right\} = \mu\vec{j}$$

$$\phi = \frac{\vec{\nabla}\cdot\vec{A}}{-i\omega\epsilon\mu}$$

$$\nabla^2\vec{A} + \omega^2\epsilon\mu\vec{A} = -\mu\vec{j}$$

$\vec{i}$ component

$$\nabla^2 A_1 + \underbrace{\omega^2\epsilon\mu A_1}_{} = -\mu j_1$$

if this were not here, Poisson equation.

$e^{i\omega t}$

$$i\omega = \frac{\partial}{\partial t}$$

$$\omega^2 = -\frac{\partial^2}{\partial t^2}$$

$$\nabla^2 A_1 - \epsilon\mu\frac{\partial^2 A_1}{\partial t^2} = -\mu j_1$$

$$\frac{\partial^2 A_1}{\partial t^2} = \frac{1}{\epsilon\mu}\nabla^2 A_1 + \frac{\mu}{\epsilon\mu}j_1$$

non-homogeneous wave equation.

Curves & line integrates.

Parametrize a curve:

straight line:

"time" or dummy parameter

$$\vec{R} = \vec{R}_0 + t\,\vec{V} \qquad -\infty < t < \infty$$

pt on line

|| to line

circle

$$\vec{R} = a\cos\theta\,\vec{i} + a\sin\theta\,\vec{j} + \vec{R}_0 \qquad 0 \le \theta \le 2\pi$$

radius

parameter

center

Parametrize a surface

$$\vec{R} = \vec{R}(s, t) \qquad \text{(it's 2-dimensional.)}$$

dummy variables

Plane:

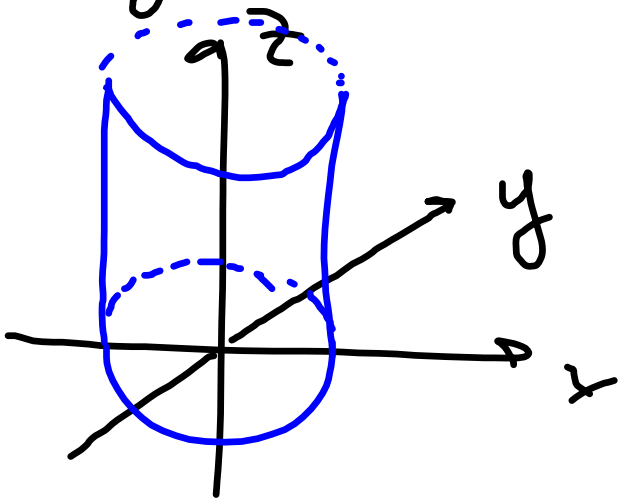
$$\vec{R} = \vec{R}_0 + s\vec{A} + t\vec{B} \qquad -\infty < s < \infty$$

$$-\infty < t < \infty$$

pt in plane

vectors in plane

Cylinder



$x = a\cos\theta$

$y = a\sin\theta$ $\qquad 0 \le \theta \le 2\pi$

$z = z$ free $\qquad 0 \le z \le b$

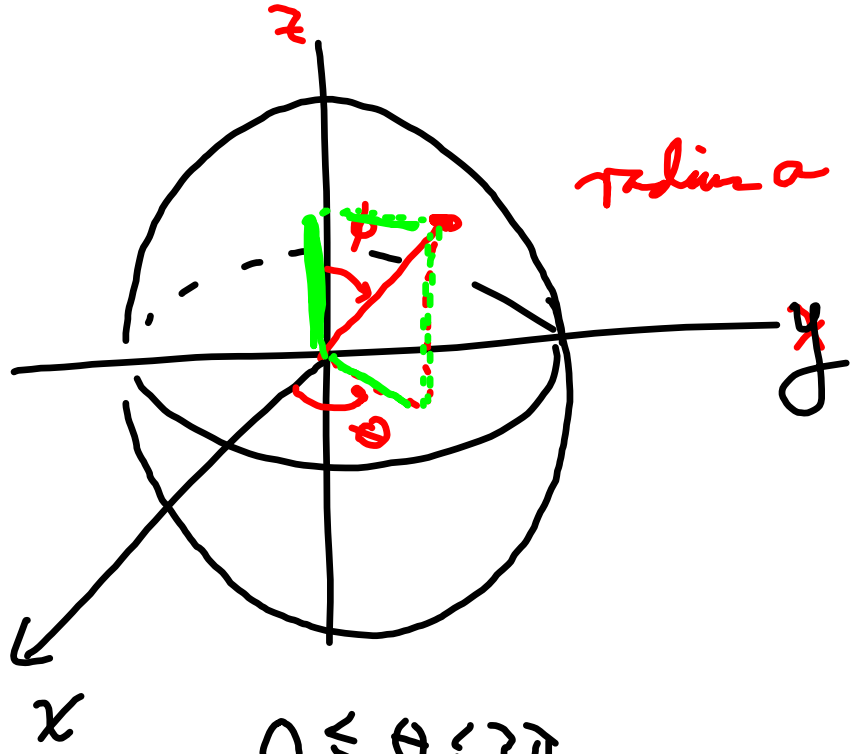
$\vec{R} = a\cos t\,\vec{i} + b\sin t\,\vec{j} + s\vec{k}$

$0 \le t \le 2\pi \qquad 0 \le s \le b$

radius $= a$

height $= b$

sphere



$$z = a\cos\phi$$

$$x = a\sin\phi\cos\theta$$

$$y = a\sin\phi\sin\theta$$

$$0 \le \theta < 2\pi$$

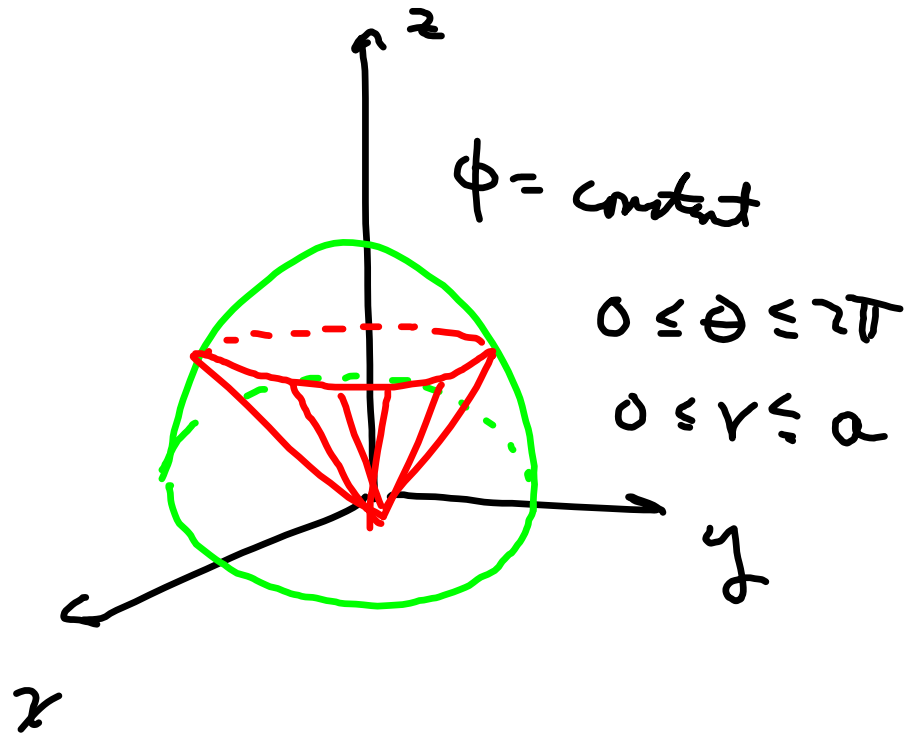
$$0 \le \phi \le \pi$$

$$\vec{R} = \vec{R}(s,t) = a\sin s\cos t\,\vec{i} + a\sin s\sin t\,\vec{j} + a\cos s\,\vec{k}$$

$$s \longleftrightarrow \phi$$

$$t \longleftrightarrow \theta$$

Cone



$\phi =$ constant

$0 \le \theta \le 2\pi$

$0 \le r \le a$

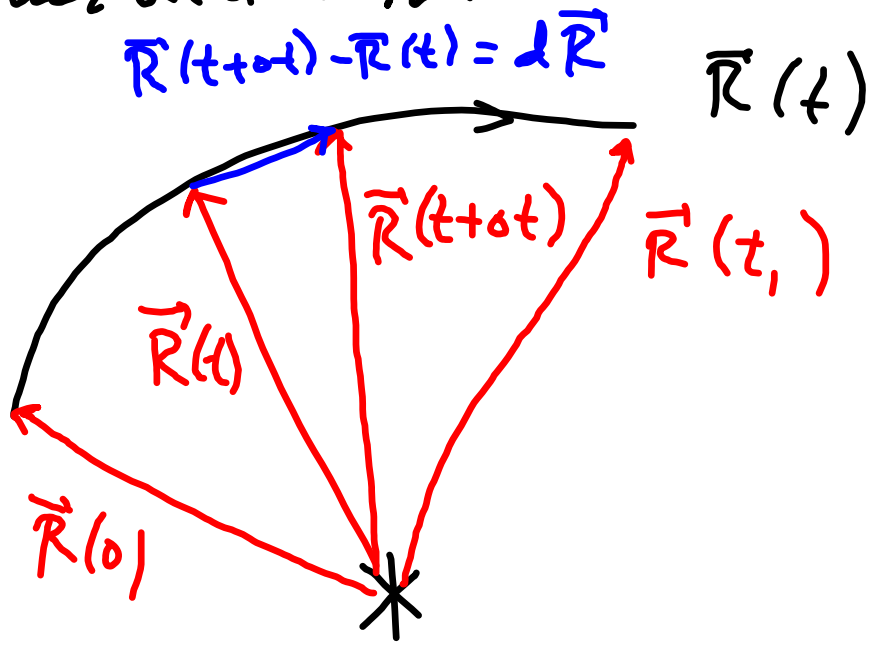
$$\vec{R} = \vec{R}(s,t) = \overset{r}{\cancel{a}} \sin \overset{\phi_0}{\cancel{\phi}} \cos t \, \vec{i}$$
$$+ \overset{r}{\cancel{a}} \sin \overset{\phi_0}{\cancel{\phi}} \sin t \, \vec{j}$$
$$+ \overset{r}{\cancel{a}} \cos \underset{\phi_0}{\cancel{\phi}} \, \vec{k}$$

$$0 \le t \le 2\pi$$
$$0 \le r \le a$$

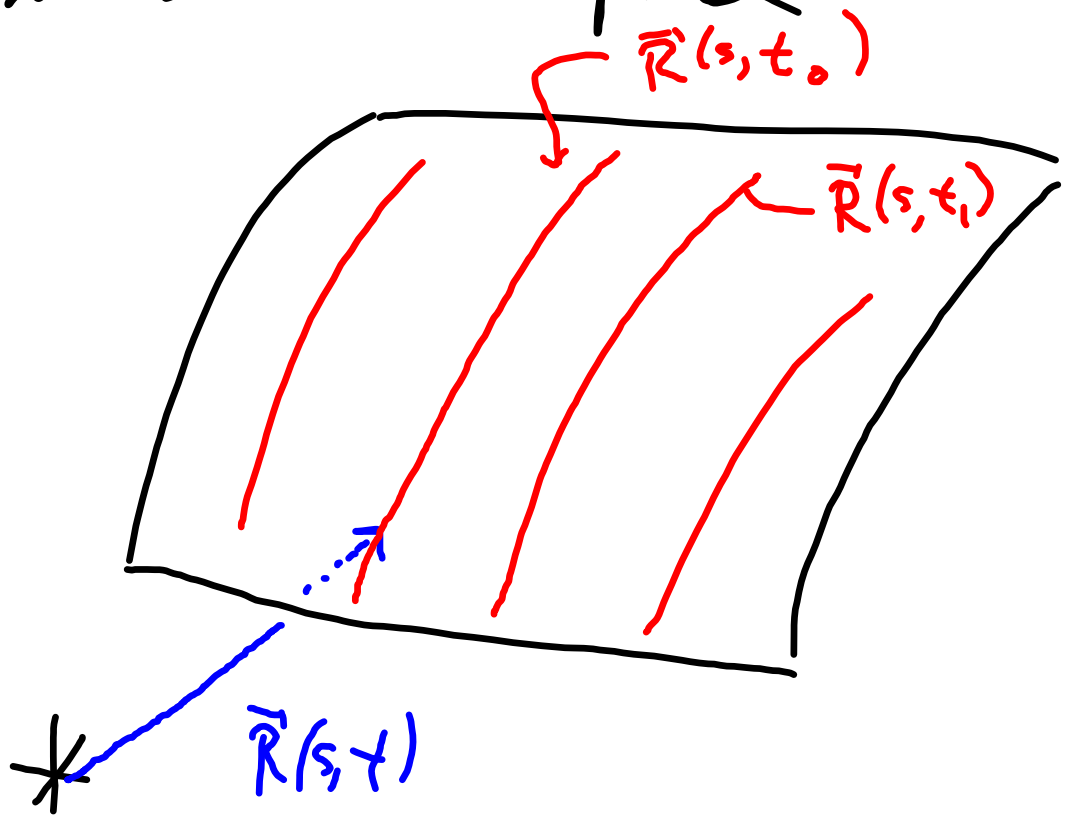
$$\cancel{s \longleftrightarrow \phi}$$
$$\cancel{t \longleftrightarrow \theta}$$

$$\vec{R} = \vec{R}(r,t)$$

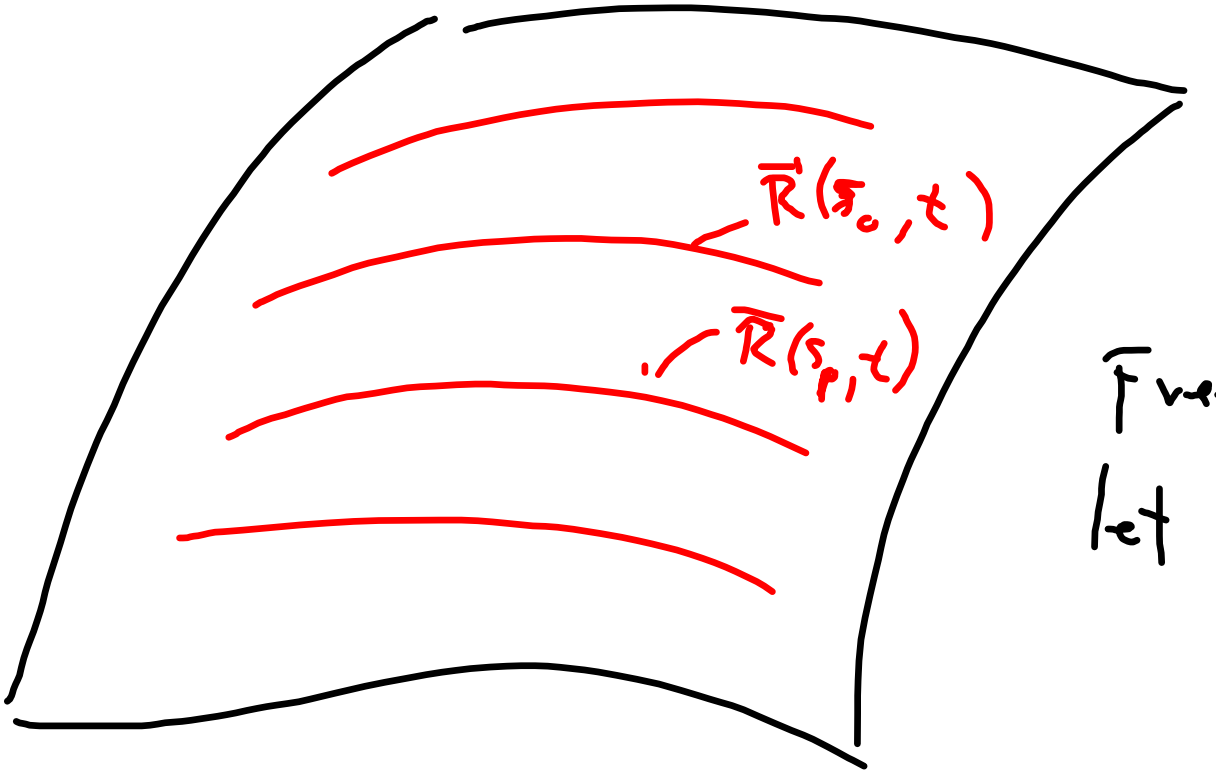
Calculus on a curve



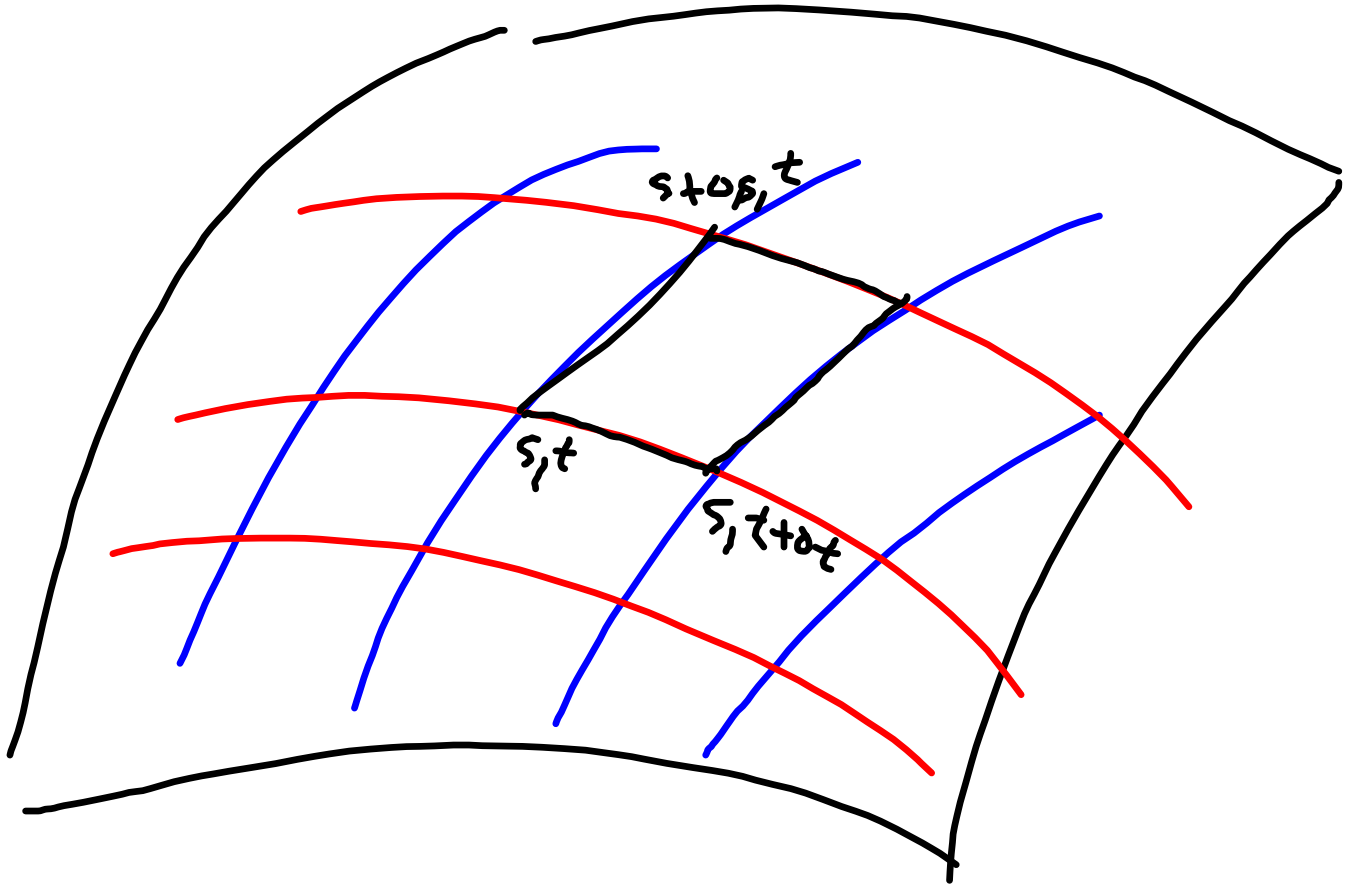
Calculus on a surface



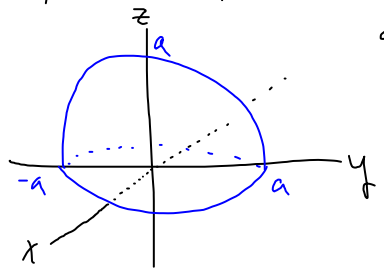
Freeze t and let s vary.



Fix s,
let t vary


s+Δs, t
s, t
s, t+Δt

Area of a hemisphere



$$\text{area} = \iint \left| \frac{\partial \vec{R}}{\partial u} \times \frac{\partial \vec{R}}{\partial v} \right| du\,dv$$

$$\vec{R} = x\,\vec{i} + y\,\vec{j} + z\,\vec{k}$$

$$= a\sin\phi\cos\theta\,\vec{i} + a\sin\phi\sin\theta\,\vec{j} + a\cos\phi\,\vec{k}$$

$$= \vec{R}(\theta, \phi)$$

$$\frac{\partial \vec{R}}{\partial \theta} \times \frac{\partial \vec{R}}{\partial \phi} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -a\sin\phi\sin\theta & a\sin\phi\cos\theta & 0 \\ +a\cos\phi\cos\theta & a\cos\phi\sin\theta & -a\sin\phi \end{vmatrix}$$

$$= \vec{i}\left[-a^2\sin^2\phi\cos\theta\right] + \vec{j}\left[-a^2\sin^2\phi\sin\theta\right]$$

$$+ \vec{k}\left[-a^2\sin\phi\cos\phi\sin^2\theta - a^2\sin\phi\cos\phi\cos^2\theta\right]$$

$$\left| \frac{\partial \vec{R}}{\partial \theta} \times \frac{\partial \vec{R}}{\partial \phi} \right| = a^2 \sqrt{\underbrace{\sin^2\phi\cos^2\theta + \sin^2\phi\sin^2\theta}_{\sin^2\phi\left[\cos^2\theta + \sin^2\theta\right]} + \cos^2\phi}\,\cdot\sin\phi$$

$$= a^2\sin\phi$$

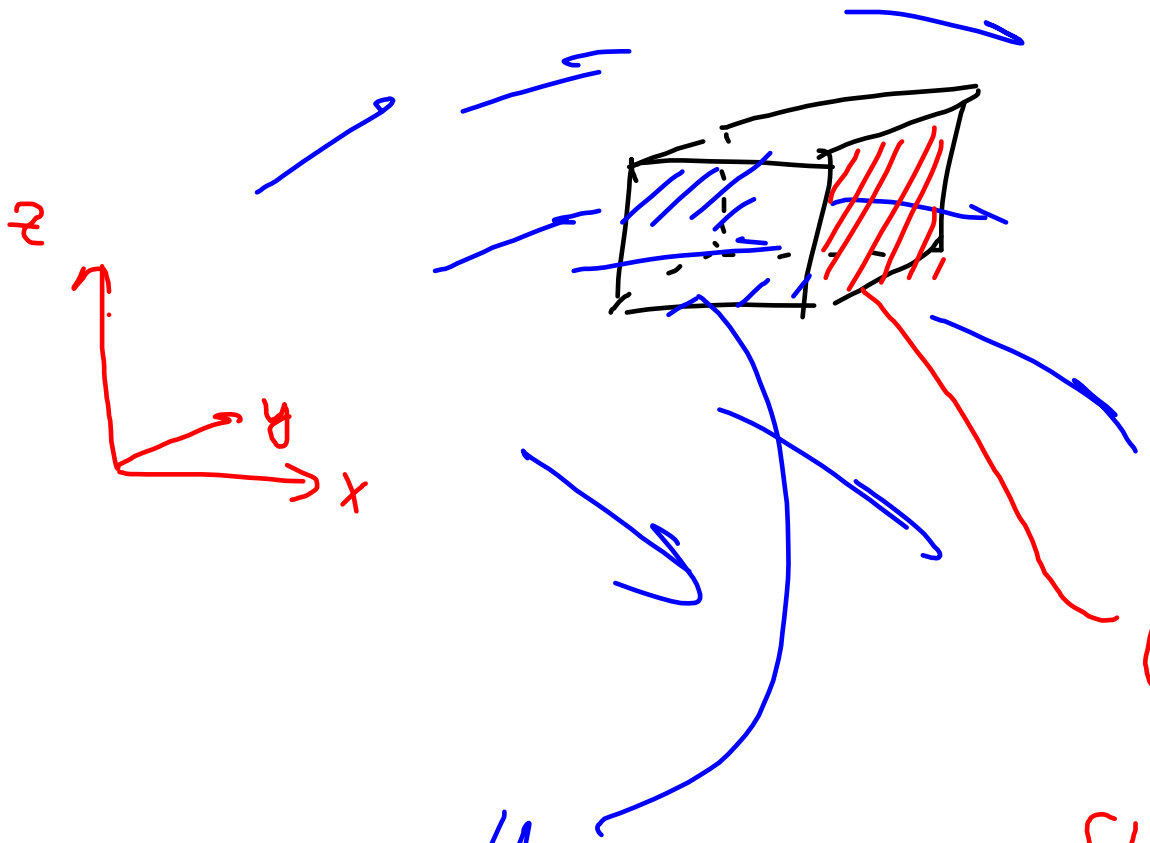
We have to evaluate

$$\int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi/2} a^2 \sin\phi \, d\theta \, d\phi$$

$$= a^2 \, 2\pi \underbrace{\Big[-\cos\phi\Big]_0^{\pi/2}}_{[-0-(-1)]}$$

$$= 2\pi a^2$$

Flux integrals.

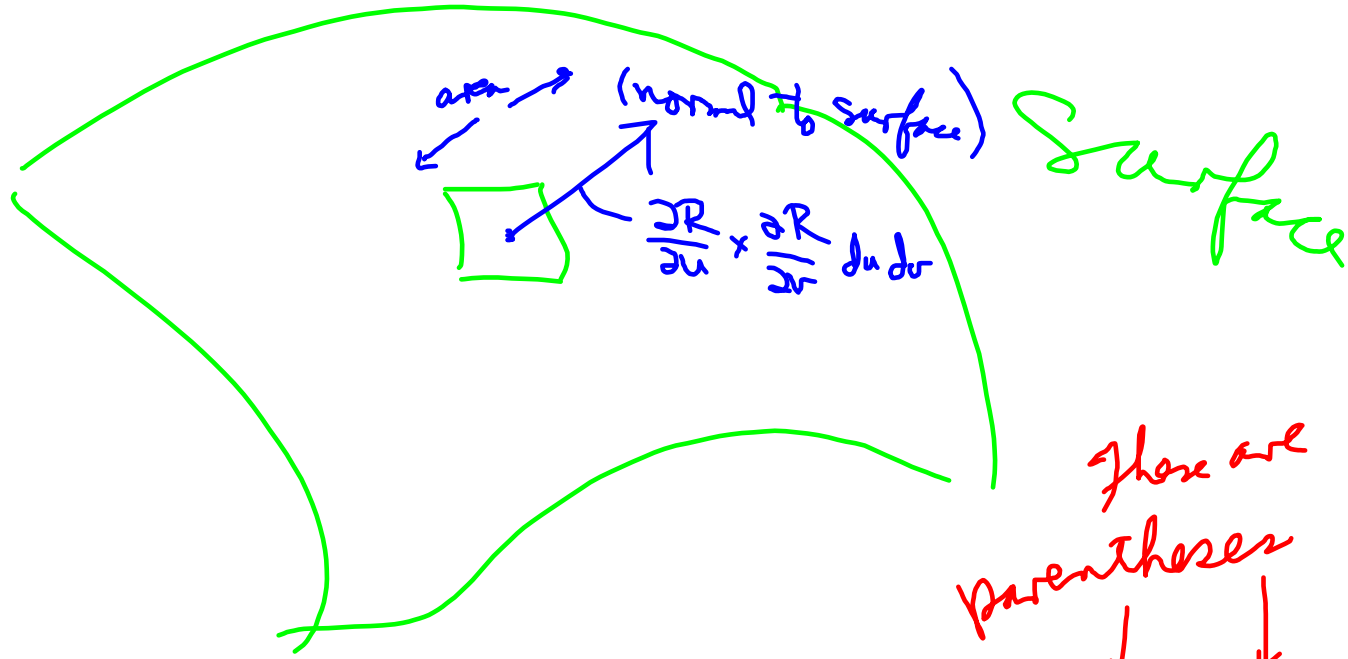


face $x =$ constant

area $= \Delta y \, \Delta z$

flux $= F_1 \, \Delta y \, \Delta z$

flux $= F_2 \, \Delta x \, \Delta z \, (\pm)$



Flux integral $\displaystyle\iint_{\text{surface}} F_n \cdot \text{area} = \iint \vec{F} \cdot \left(\frac{\partial \vec{R}}{\partial u} \times \frac{\partial \vec{R}}{\partial v} \right) du\, dv$

These are parentheses

$$= \iint \underbrace{|\vec{F}| \cos\theta}_{\vec{F},\,\text{normal}} \, \underbrace{\left| \frac{\partial \vec{R}}{\partial u} \times \frac{\partial \vec{R}}{\partial v} \right|}_{\text{area}} du\, dv$$

Example surface:

$$x = u + v \qquad 0 \le u \le 1$$
$$y = u - v \qquad 0 \le v \le 1$$
$$z = u^2$$

$$\vec{R} = (u+v)\vec{i} + (u-v)\vec{j} + u^2\vec{k}$$

$$\vec{F} = \vec{F}(x,y,z) = \vec{i} + xy\,\vec{j}$$

Evaluate $\iint \vec{F}\cdot d\vec{S}$

$d\vec{S}$ means $\frac{\partial \vec{R}}{\partial u} \times \frac{\partial \vec{R}}{\partial v}\, du\, dv$

$= \vec{n}_{\text{unit normal}} \cdot d(\text{area})$

$$= \int_u \int_v \vec{F} \cdot \frac{\partial \vec{R}}{\partial u} \times \frac{\partial \vec{R}}{\partial v}\, du\, dv$$

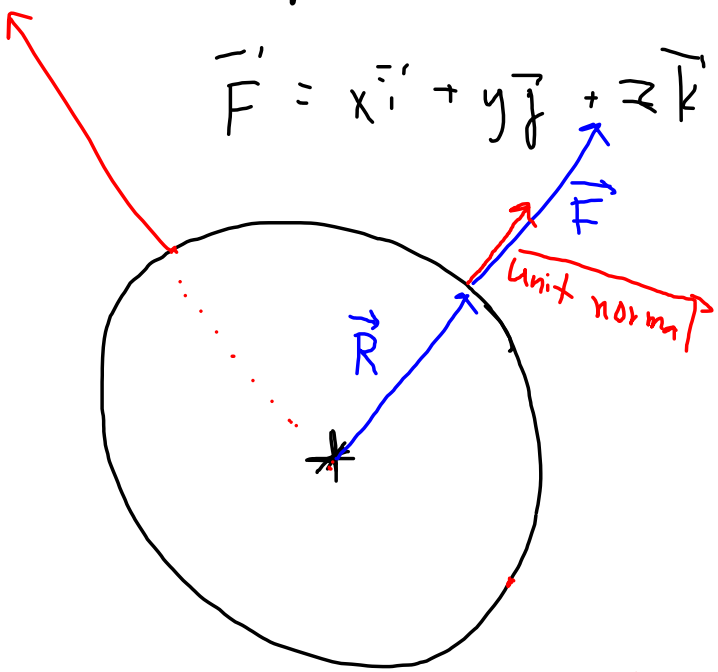
$$dS = d(\text{area}) = |d\vec{S}|$$

$$= \int_u \int_v \begin{vmatrix} F_1 & F_2 & F_3 \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix} du\, dv = \int_{u=0}^{1} \int_{v=0}^{1} \begin{vmatrix} 1 & \overset{(u+v)(u-v)}{\cancel{xy}} & 0 \\ 1 & 1 & 2u \\ 1 & -1 & 0 \end{vmatrix} du\, dv$$

Example 4.21

surface: sphere of radius 2

$$\vec{F} = x\vec{i} + y\vec{j} + z\vec{k} = \vec{R}$$



$$\iint \vec{F}\cdot d\vec{S}$$

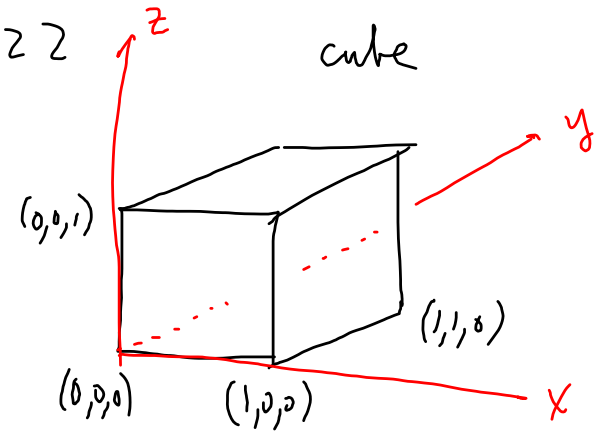
$$= \iint \vec{F}\cdot\vec{n}\, d\text{area}$$

$$= \iint 2\, (d\text{area}) = 2 \times \text{total area}$$

$$= 2 \times 4\pi \cdot 2^2$$

$$\vec{F}\cdot\vec{n} = |\vec{F}|\,|\vec{n}|\cos\theta$$
$$= 2 \cdot 1 \cdot 1$$

Example 4.22 cube



$\vec{F} = \vec{R} = x\vec{i} + y\vec{j} + z\vec{k}$

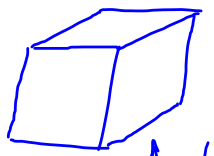
$\iint \vec{F} \cdot d\vec{S}$

$\Rightarrow$ unit normal is $\vec{i}$

$\vec{F} \cdot \vec{n}_{unit} = x = 1 = \text{constant}$

area $= 1 \times 1$

flux $= 1 \cdot 1 \cdot 1 = 1$



bottom flux $= \cancel{\iint (y\vec{j})(-\vec{j})\, d\text{area}}$

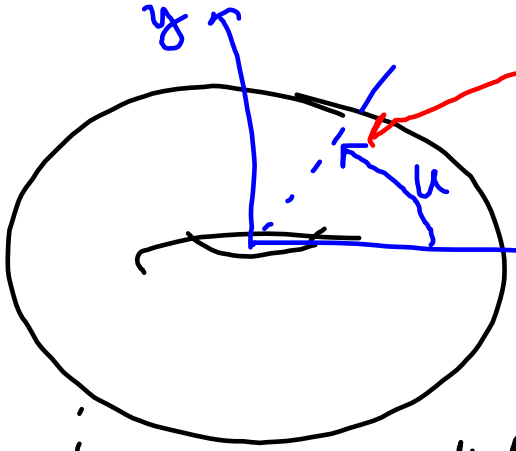
$= \cancel{\iint 0 \cdot dx\, dy = 0}$

normal $= -\vec{k}$

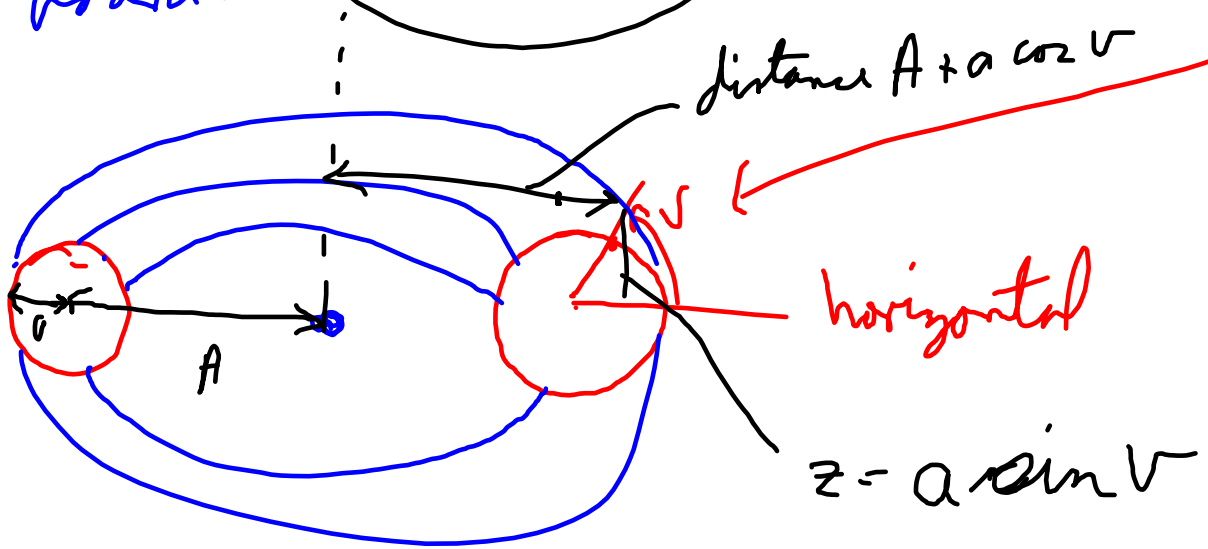
$\vec{F} \cdot \vec{n} = (z\vec{k})(-\vec{k}) = -z = 0$

Torus

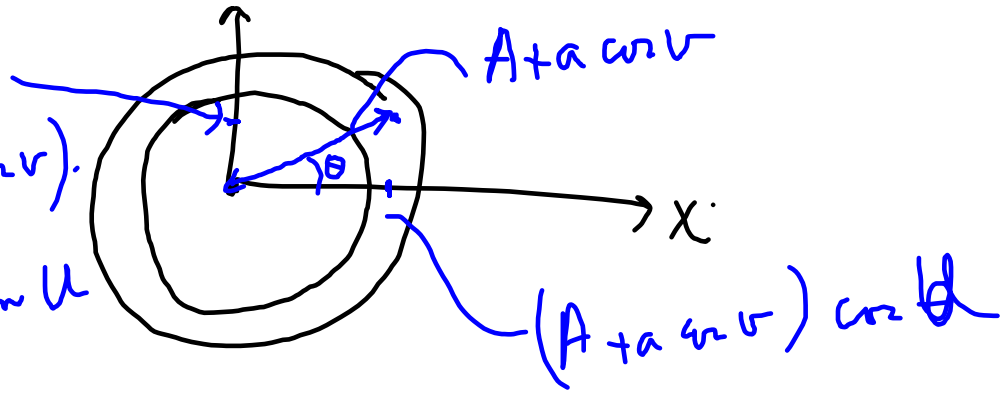
z, out of the board.



Look at cross section



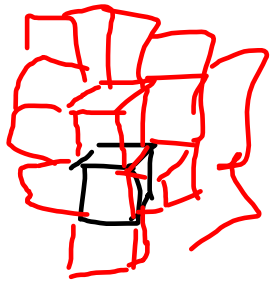
From the top



Divergence Theorem.

$$\boxed{\iiint_{\text{volume}} \vec{\nabla}\cdot\vec{F}\,dx\,dy\,dz = \iint_{\text{enclosing surface}} \vec{F}\cdot d\vec{S}}$$

divergence = outflux per unit volume



$\sum_{\text{all cubes}}$ outflux

= net outflux through bndry faces.

$\sum$ outfluxes = outflux out the exterior (interior fluxes cancel).

$$d\vec{S} = \frac{\partial \vec{R}}{\partial u} \times \frac{\partial \vec{R}}{\partial v}\,du\,dv$$

$$= \vec{n}_{\text{unit normal}}\, d(\text{area})$$

outward

Prob. 4 Final 03a.

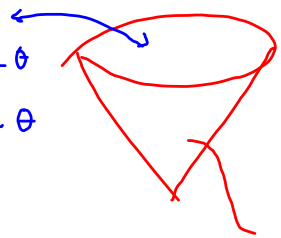
$x = \rho\cos\theta$

$y = \rho\sin\theta$

$z = 2$

$0 \le \theta \le 2\pi$

$0 \le \rho \le 2$



$z = r$

$x = r\cos\theta$

$y = r\sin\theta$

$0 \le r \le 2$

$0 \le \theta \le 2\pi$

$$\iint \vec{F}\cdot d\vec{S} = \iiint \vec{\nabla}\cdot\vec{F}\, d\text{volume}$$

$$\vec{F} = y^2\vec{i} + z\vec{k}$$

$$\vec{\nabla}\cdot\vec{F} = 1$$

$$= \iiint 1\, d\text{volume}$$

$$= \frac{1 \times \pi \times 2^2 \times 2}{3}$$

Lid: $\vec{R}_L = \rho\cos\theta\,\vec{i} + \rho\sin\theta\,\vec{j} + 2\vec{k}$ (blue)

Cone $\vec{R}_C = r\cos\theta\,\vec{i} + r\sin\theta\,\vec{j} + r\,\vec{k}$ (red)

$$\text{Lid outflow} = \int_{\rho=0}^{2}\int_{\theta=0}^{2\pi} \vec{F}\cdot \frac{\partial \vec{R}_L}{\partial \rho} \times \frac{\partial \vec{R}_L}{\partial \theta}\, d\theta\, dr$$

$$+ \text{ Cone outflow} = \int_{r=0}^{2}\int_{\theta=0}^{2\pi} \vec{F}\cdot \frac{\partial \vec{R}_C}{\partial r} \times \frac{\partial \vec{R}_C}{\partial \theta}\, d\theta\, dr$$

$$8\pi/3$$

Final 02b #7.

On top $\vec{n} = \vec{k}$, $\vec{F} \cdot \vec{n} = z = 2$,

$$\iint \vec{F} \cdot \vec{n} \, d\text{area} = 2 \cdot \pi \cdot 1^2 = 2\pi$$

On bottom $\vec{n} = -\vec{k}$, $\vec{F} \cdot \vec{n} = -z = -1$

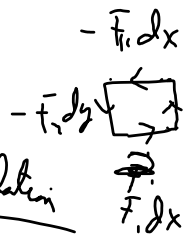
$$\iint \vec{F} \cdot \vec{n} \, dS = \iint \vec{F} \cdot \vec{n} \, d\text{area} = (-1)\pi 1^2 = -\pi$$

$$\text{Top} + \text{Bottom} + \text{Side Flux} = \iiint \underbrace{\nabla \cdot \vec{F}}_{0} \, d\text{volume}$$

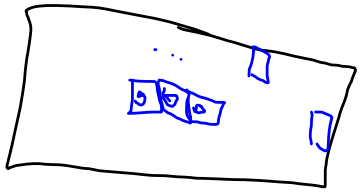
$$\therefore \text{Side flux} = -\pi$$

Stokes' Theorem.

$$\vec{\nabla} \times \vec{F} = \frac{\text{circulation}}{\text{area}}$$



$$= \int \vec{F} \cdot d\vec{R} = \text{circulation around}$$



$$\iint (\vec{\nabla} \times \vec{F}) \cdot \vec{n} \, d\,\text{area} = \oint \vec{F} \cdot d\vec{r}$$

boundary of curve

$$\iint_{\text{surface}} (\vec{\nabla} \times \vec{F}) \cdot d\vec{S} = \oint_{\text{rim of surface}} \vec{F} \cdot d\vec{r}$$

$$d\vec{S} = \frac{\partial \vec{R}}{\partial u} \times \frac{\partial \vec{R}}{\partial v} \, du \, dv$$

$$= \vec{n} \, d\,\text{area}$$

$$\iiint \vec{\nabla}\cdot\vec{F}\, d\,\text{volume} = \iint \vec{F}\cdot d\vec{S}$$

$$\iint \vec{\nabla}\times\vec{F}\cdot d\vec{S} = \int \vec{F}\cdot d\vec{r}$$

$$\int \vec{\nabla}\phi\cdot dr = \phi(\text{end}) - \phi(\text{start})$$

Don't think about this.

Div. Theorem. $\iiint \vec{\nabla}\cdot(\vec{\nabla}\times\vec{G})\, d\,\text{volume} = \iint (\nabla\times\vec{G})\cdot d\vec{S}$

$$\shortparallel$$
$$0$$

Go to page 50 of this package for notes on Lecture 24, Curvilinear Integrals.
Return here for Lecture 25, Maxwell's Equastions 2.

Solid Angle:

inside a solid, 4π

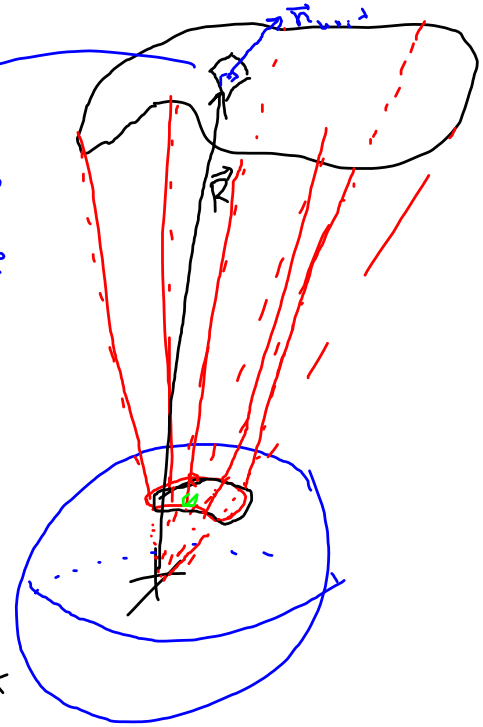
plane 2π

cube corner $\frac{1}{8} \times 4\pi = \pi/2$

First project this area to be $\perp$ to line of sight:

$$\vec{dS} \cdot \frac{\vec{R}}{|\vec{R}|}$$

(area $\hat{n}_{unit}$)



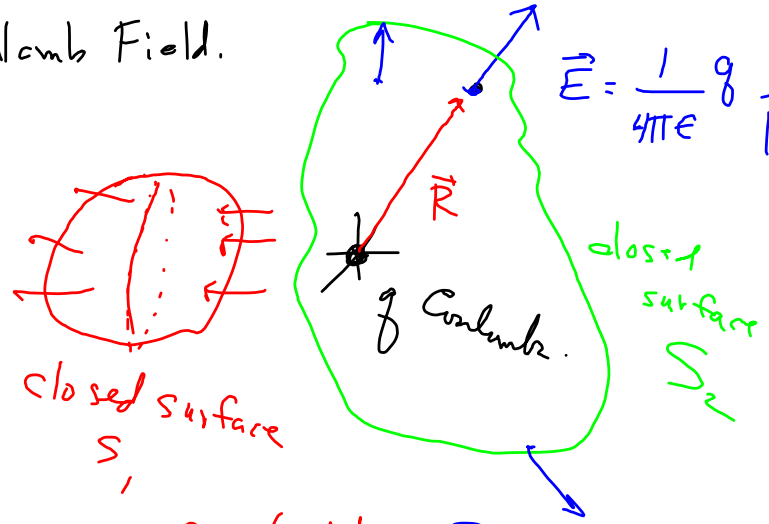
$\vec{dS}$

Project this back to <u>unit</u> sphere by dividing by $|\vec{R}|^2$

$$d\Omega = \frac{\vec{dS} \cdot \vec{R}}{|\vec{R}|^3}$$

$$\text{Total solid angle} = \iint_{\text{surface}} \frac{\vec{dS} \cdot \vec{R}}{|\vec{R}|^3}$$

Coulomb Field.



$$\vec{E} = \frac{1}{4\pi\epsilon} q \frac{\vec{R}}{|\vec{R}|^3}$$

$$\text{flux} = \frac{q}{4\pi\epsilon} \cdot (\text{solid angle} - \text{solid angle}) = 0$$

$$E\cdot\text{flux} = \iint_{S_2} \vec{E}\cdot d\vec{S}$$

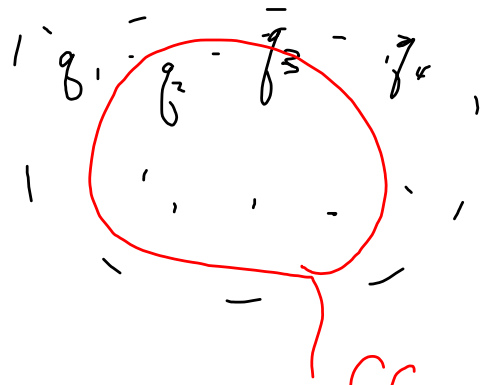
$$= \frac{1}{4\pi\epsilon} q \iint \frac{\vec{R}}{|\vec{R}|^3}\cdot d\vec{S}$$

$$= \frac{q}{4\pi\epsilon}(\text{solid angle})$$

$$= \frac{q}{4\pi\epsilon} 4\pi \quad \text{for } S_2$$

$$= \frac{q}{\epsilon}$$

Lots of charges



$\iint \vec{E} \cdot d\vec{S}$ = sum of $\frac{q}{\epsilon}$ for the charges inside

Gauss's Law: $\iint \underset{\epsilon \vec{E}}{\vec{D}} \cdot d\vec{S}$ = charge enclosed

$= \iiint \rho \, d\,\text{volume}$

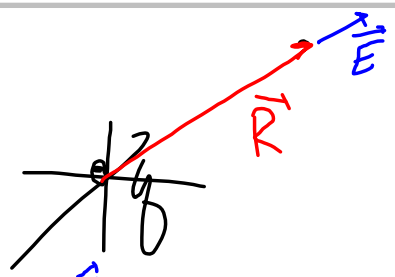
ρ = charge density

Div. Thm: flux int $= \iiint \vec{\nabla} \cdot \vec{D} \, dvol = \iiint \rho \, d\,\text{volume}$

True for any volume, so

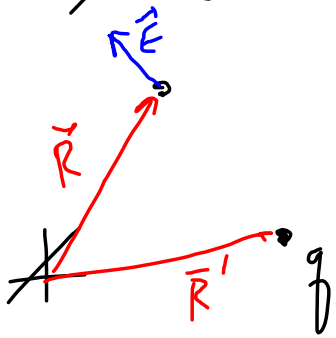
$$\boxed{\vec{\nabla} \cdot \vec{D} = \rho}$$

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{q\,\vec{R}}{|\vec{R}|^3}$$



If fixed at $\vec{R}'$,

$$\vec{E} = \frac{1}{4\pi\epsilon_0} q \frac{\vec{R}-\vec{R}'}{|\vec{R}-\vec{R}'|^3}$$



more than one charge

$$\vec{E} = \text{sum}\ \frac{1}{4\pi\epsilon_0} q_j \frac{(\vec{R}-\vec{R}_j')}{|\ \ |^3}$$

if the charge is distributed with a density ρ coulomb per unit volume

$$\vec{E}(\vec{R}) = \frac{1}{4\pi\epsilon} \iiint \frac{\rho(\vec{R}')(\vec{R}-\vec{R}')}{|\vec{R}-\vec{R}'|^3}\, d\,vol$$

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{q\,\vec{R}}{|\vec{R}|^3}$$

$$\vec{E} = -\frac{1}{4\pi\epsilon_0} q \nabla \frac{1}{|\vec{R}|}$$

$\vec{E}$ is a gradient, so its curl is zero $\vec{\nabla} \times \vec{E} = \vec{0}$

The potential $\vec{E} = -\nabla V$ $\quad V = \frac{q}{4\pi\epsilon_0 |\vec{R}|}$

If q is at $\vec{R}'$, $V = \frac{q}{4\pi\epsilon |\vec{R} - \vec{R}'|}$

For a distribution of charge,

$$V(\vec{R}) = \frac{1}{4\pi\epsilon} \iiint \frac{\rho(\vec{R}')}{|\vec{R} - \vec{R}'|} d\mathrm{vol}$$

$$\nabla \cdot \vec{E} = \rho/\epsilon \qquad \vec{E} = -\nabla V$$

$$\boxed{\nabla^2 V = -\rho/\epsilon}$$

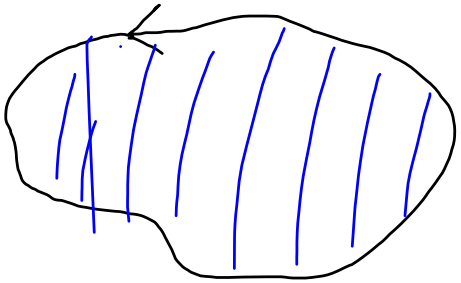
We know a solution:

$$V = \frac{1}{4\pi\epsilon} \iiint \rho(\vec{R}')/|\vec{R}-\vec{R}'| \, d\mathrm{vol}$$

if we know ρ everywhere.

Math Fact: a solution to $\nabla^2 \psi = f$ is

$$\psi = -\frac{1}{4\pi} \iiint \frac{f(\vec{R}')}{|\vec{R}-\vec{R}'|} \, d\mathrm{vol}$$



$$\int_{\text{wire}} \vec{E}\cdot d\vec{R} = -V_{\text{potential}}$$

Faraday $\frac{d}{dt}\iint_{\substack{\text{surface}\\ \text{bounded by}\\ \text{wire}}} \vec{B}\cdot d\vec{S} = +V_{\text{voltage}}$

$$= -\int_{\text{wire}} \vec{E}\cdot d\vec{R}$$

Stokes' Thm $\quad -\iint \vec{\nabla}\times\vec{E}\cdot d\vec{S}$

any surface $\quad \frac{d}{dt}\iint \vec{B}\cdot d\vec{S} = -\iint \vec{\nabla}\times\vec{E}\cdot d\vec{S}$

$$\iint \frac{\partial \vec{B}}{\partial t}\cdot d\vec{S} = \text{———}$$

So $\quad \frac{\partial \vec{B}}{\partial t} = -\vec{\nabla}\times\vec{E} \quad$ Faraday's law

In appendix D of Davis & Snider,

$$\text{"}K\text{"} = \frac{1}{4\pi\epsilon} \qquad \text{"}\gamma\text{"} = \frac{\mu}{4\pi} \qquad \text{"}n\text{"} = 1$$

Maxwell's Equations

$$\nabla \cdot \vec{D} = \rho \qquad \vec{D} = \epsilon \vec{E}$$

$$\nabla \cdot \vec{B} = 0 \qquad \vec{B} = \mu \vec{H}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = -i\omega \vec{B}$$

conservation law

time factor $e^{i\omega t}$

$$\nabla \times \vec{B} = \mu \vec{J} + \mu \frac{\partial \vec{D}}{\partial t} = \mu \vec{J} + i\omega\mu \vec{D}$$

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \vec{J}$$

$$\frac{\partial \rho}{\partial t} = i\omega\rho$$

Take $\nabla \cdot$ fourth equation. You get first eq. (redundant)

Take $\nabla \cdot$ third eq. You ~~ge~~ get second eq. (redundant)

$$\nabla\cdot\vec{D} = \rho$$

$$\nabla\cdot\vec{B} = 0 \longrightarrow \text{so } \vec{B} = \vec{\nabla}\times\vec{A}$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} = -i\omega\vec{B} \rightarrow \vec{\nabla}\times\vec{E} + i\omega\nabla\times\vec{A} = 0$$

time factor $e^{i\omega t}$

$$\vec{\nabla}\times\vec{B} = \mu\vec{J} + \mu\frac{\partial\vec{D}}{\partial t} = \mu\vec{J} + i\omega\mu\vec{D}$$

$$\nabla\times\left[\vec{E} + i\omega\vec{A}\right] = 0$$

So

$$\vec{E} + i\omega\vec{A} = -\nabla\phi$$

$$\vec{E} = -\nabla\phi - i\omega\vec{A}$$

4th eq. $\vec{\nabla}\times[\vec{\nabla}\times\vec{A}]$

$$\vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A} = \mu\vec{J} + i\omega\mu\epsilon\left[-\nabla\phi - i\omega\vec{A}\right]$$

$$\vec{\nabla}\left[\vec{\nabla}\cdot\vec{A} + i\omega\mu\phi\epsilon\right] - \vec{\nabla}^2\vec{A} = \mu\vec{J} + \omega^2\epsilon\mu\vec{A}$$

Exploit the slack in $\vec{A}$ to enforce $\nabla\cdot\vec{A} + i\omega\mu\epsilon\phi = 0$

$$\vec{\nabla}^2\vec{A} = -\mu\vec{J} + \omega^2\epsilon\mu\vec{A}$$

If $\omega = 0$ $\quad \vec{\nabla}^2\vec{A} = -\mu\vec{J}$

$$\nabla^2 A_1 = -\mu j_1$$

$$\nabla^2 A_2 = -\mu j_2$$

$$\nabla^2 A_3 = -\mu j_3$$

From page 7, a solution is

$$\vec{A} = +\frac{\mu}{4\pi}\iiint\frac{\vec{J}(\vec{R}')}{|\vec{R}-\vec{R}'|}\,dvol$$

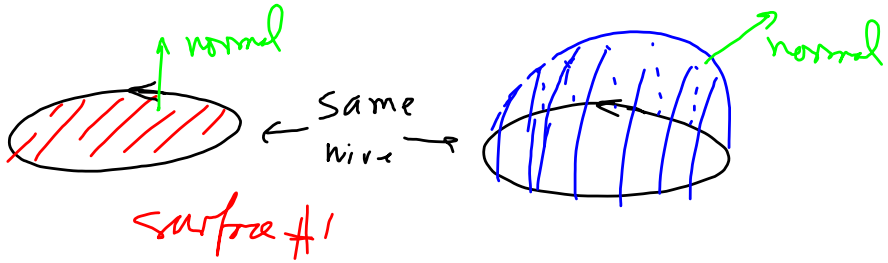
If $\omega \neq 0$, sol'n is

$$\vec{A} = +\frac{\mu}{4\pi}\iiint\frac{\vec{J}(\vec{R}')}{|\vec{R}-\vec{R}'|}e^{i|\vec{R}-\vec{R}'|\omega\sqrt{\epsilon\mu}}\,d\text{volume}$$

Faraday

"Magnetic Flux" $= \iint_S \vec{B} \cdot d\vec{S}$

$$\frac{d}{dt} \iint_{\text{surface}} \vec{B} \cdot d\vec{S} = \oint_{\text{wire}} \vec{E} \cdot d\vec{r}$$



It better be true that $\iint_{S\#1} \vec{B}\cdot d\vec{S} = \iint_{S\#2} \vec{B}\cdot d\vec{S}$.

Solid

outflux of $\vec{B} = \iint_{\text{whole surface}} \vec{B}\cdot d\vec{S}_{\text{outer}} = \iint \vec{B}\cdot d\vec{S}_{\text{out}} + \iint \vec{B}\cdot d\vec{S}_{\text{out}}$

Does $\iint_{\text{solid}} \vec{B}\cdot d\vec{S} = 0$? This is Faraday flux #1

This is (-) Fa. Flux #2

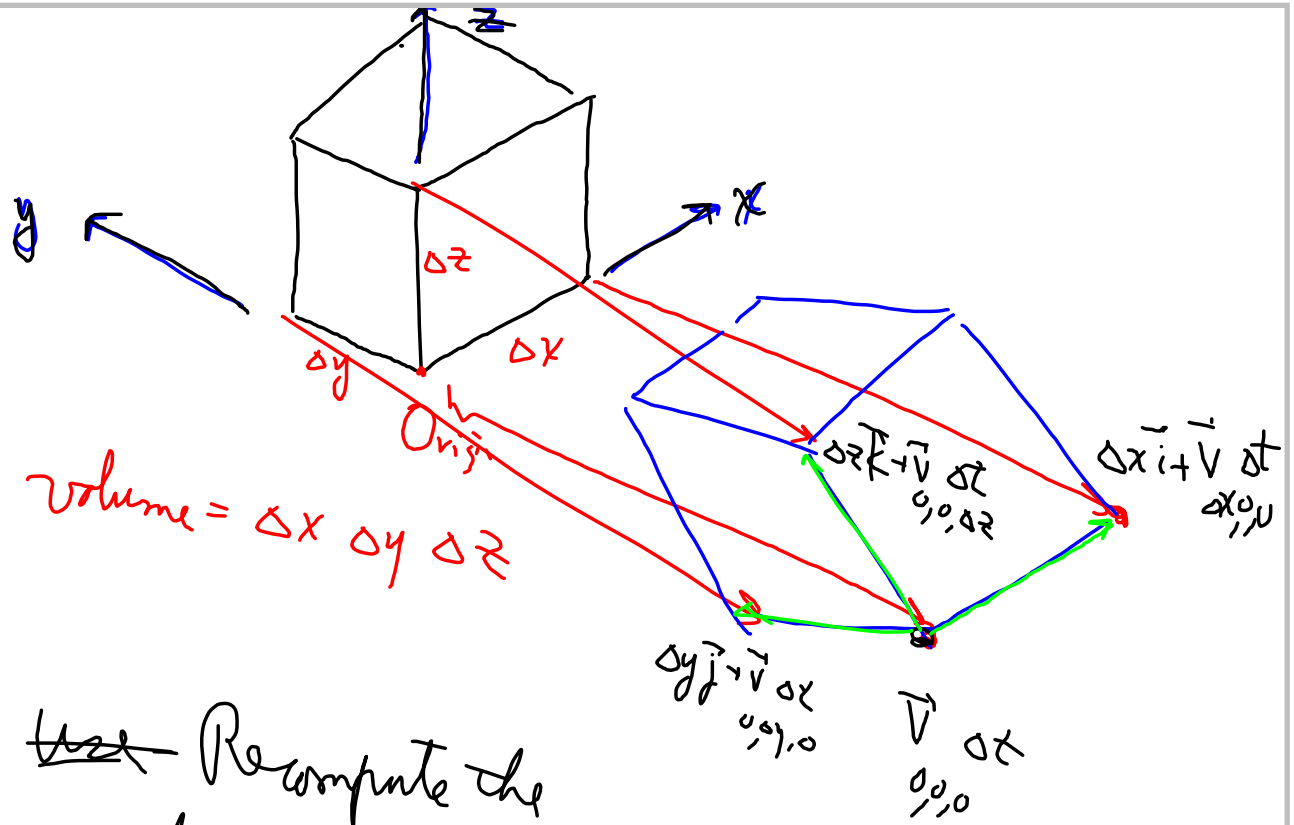
Div. Thm. $\iint_{\text{solid}} \vec{B}\cdot d\vec{S} = \iiint \underbrace{\vec{\nabla}\cdot\vec{B}}_{=0 \text{ (Maxwell)}}\, dvolume$

Fluid-flow

Velocity field

velocity $\vec{V} = (\vec{R})$

divergence of velocity $\vec{\nabla} \cdot \vec{V} = \dfrac{d\,\text{volume}}{dt} \dfrac{1}{\text{volume}}$



volume = Δx Δy Δz

~~Use~~ Recompute the volume using the triple scalar product of the 3 sides, you will find

$$\frac{\Delta V}{V} = (\vec{\nabla}\cdot\vec{V})\,\Delta t \qquad \frac{dV}{dt}\frac{1}{V} = \nabla\cdot\vec{V}$$

Transport Theorems.

Flux transport theorem.

$$\iint \vec{B} \cdot d\vec{S}$$

$$\frac{\partial \vec{B}}{\partial t} \neq 0$$ Surface is moving

Scalar potential.

vector $\vec{F}(\vec{R}) = \nabla \phi(\vec{R})$

First check that $\vec{\nabla} \times \vec{F} = \vec{0}$

$$x\vec{i} + z\vec{j} + y\vec{k} = \nabla\left(\frac{x^2}{2} + yz\right)$$

Try

$$\phi = \int x\,dx + \int z\,dy + \int y\,dz \quad \text{wrong}$$

$$x^2/2 + zy + yz = x^2/2 + 2yz$$

$$\cancel{x\vec{i} + y}$$

$$x\vec{i} + z\vec{j} + (y+z^2)\vec{k} = \vec{\nabla}\phi$$

need $\dfrac{\partial \phi}{\partial x} = x \longleftarrow \phi = \dfrac{x^2}{2} + \ldots z \ldots$

$\dfrac{\partial \phi}{\partial y} = z \longleftarrow \phi = \dfrac{x^2}{2} + zy + \ldots z \ldots$

$\dfrac{\partial \phi}{\partial z} = y + z^2 \longleftarrow 0 + y + \cancel{z^2}$

$$\phi = \frac{x^2}{2} + yz + z^3/3$$

$$\vec{F} = x\vec{i} + z\vec{j} + (2y+z^2)\vec{k}$$

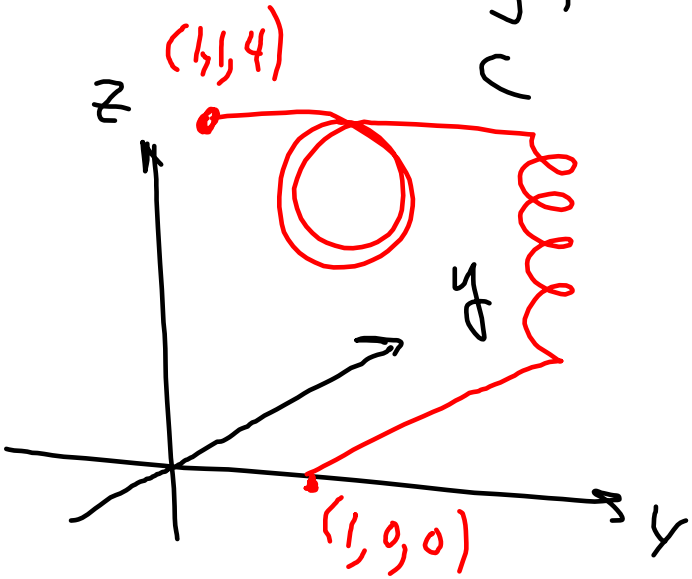
$\vec{\nabla}\times\vec{F} \neq \vec{0}$, you won't make it.

$$\phi = \frac{x^2}{2} + zy + \psi(z)$$

$$\frac{\partial \phi}{\partial z} = 2y + z^2 = 0 + y + (y+z^2)$$

Example: $\vec{F} = x\vec{i} + z\vec{j} + (y+z^2)\vec{k}$

Work out $\int_C \vec{F} \cdot d\vec{R}$



Soln:

$\vec{F} = \vec{\nabla}\left(\frac{x^2}{2}\cancel{\vec{i}} + yz\cancel{\vec{j}} + \frac{z^3}{3}\cancel{\vec{k}}\right)$

$\int \vec{F} \cdot d\vec{R} = \phi(\text{end}) - \phi(\text{start})$

$= \left(\frac{x^2}{2} + yz + \frac{z^3}{3}\right)\Big|_{(1,0,0)}^{(1,1,4)}$

Find the vector potential:

$$\vec{F}(x,y) = (x+y)\vec{i} + (-y+x)\vec{j}$$

2-D field only x, y only $\vec{i}, \vec{j}$

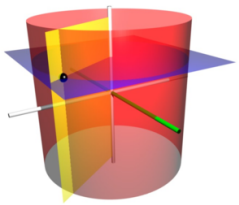
There is a vector potential of the form $0\vec{i} + 0\vec{j} + \psi\vec{k}$.

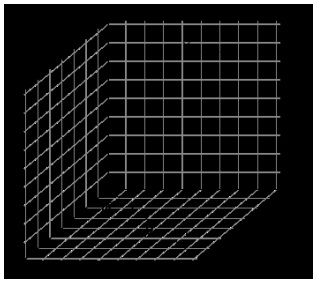
$$(x+y)\vec{i} + (-y+x)\vec{j} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ 0 & 0 & \psi \end{vmatrix}$$

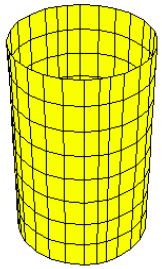
$$x+y = \frac{\partial \psi}{\partial y} \qquad \psi = xy + \frac{y^2}{2} + \dots \underbrace{x \dots}_{-x^2/2}$$

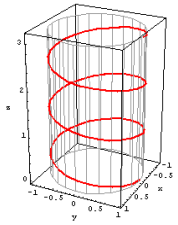
$$-y+x = -\frac{\partial \psi}{\partial x} \qquad = -y + 0$$

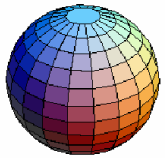
answer: $0\vec{i} + 0\vec{j} + \left(xy + \frac{y^2}{2} - \frac{x^2}{2}\right)\vec{k}$

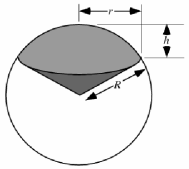


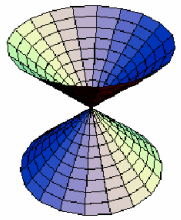








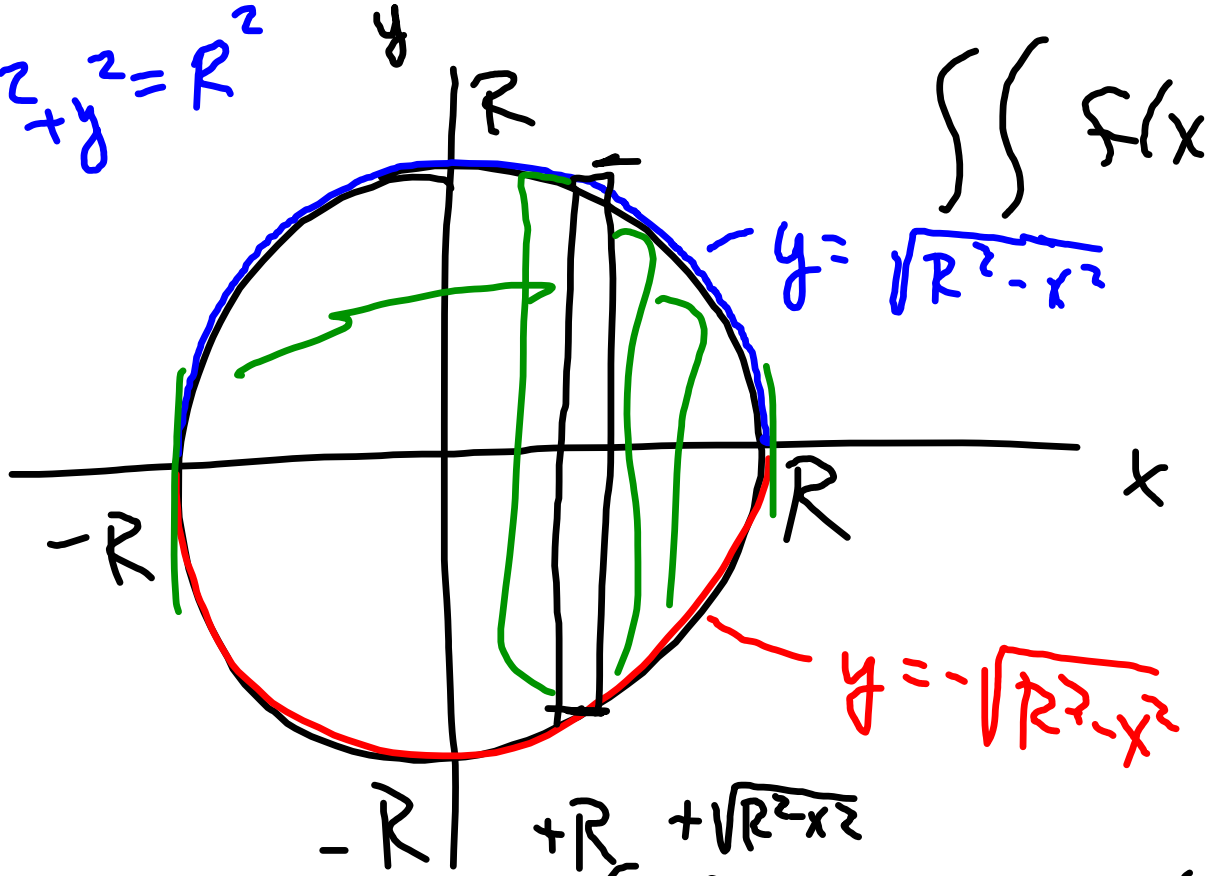




http://mathworld.wolfram.com/Sphere.html

http://mathworld.wolfram.com/Cone.html

$x^2+y^2=R^2$

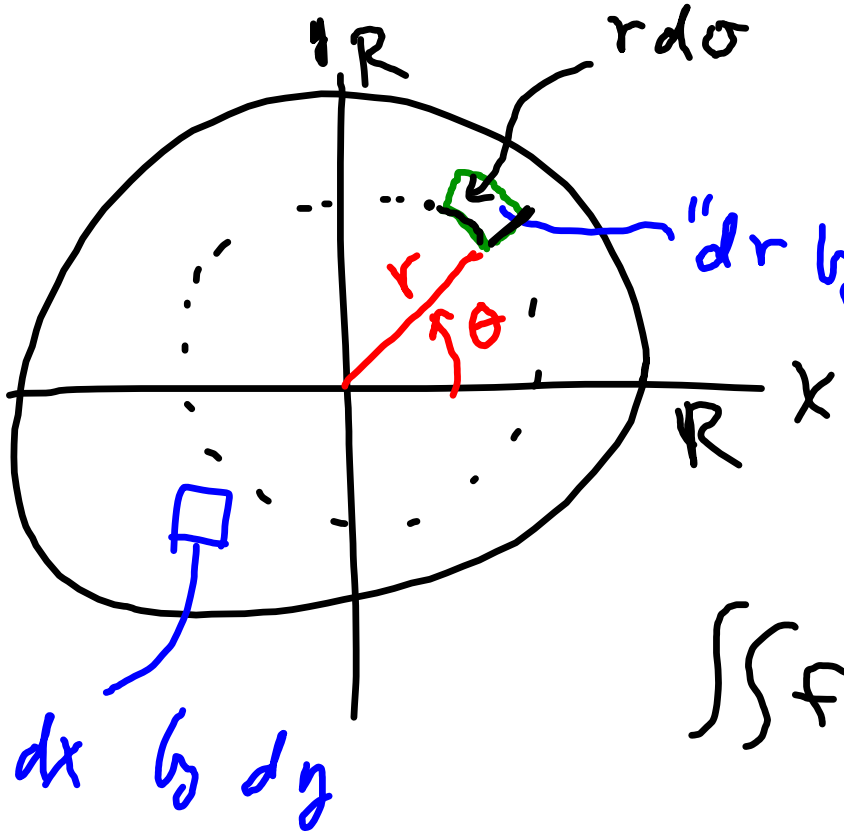


$y=\sqrt{R^2-x^2}$

$y=-\sqrt{R^2-x^2}$

$$\iint f(x,y)\,\underbrace{dx\,dy}_{\text{darea}}$$

$$\int_{x=-R}^{+R}\int_{y=-\sqrt{R^2-x^2}}^{+\sqrt{R^2-x^2}} f(x,y)\,dy\,dx$$



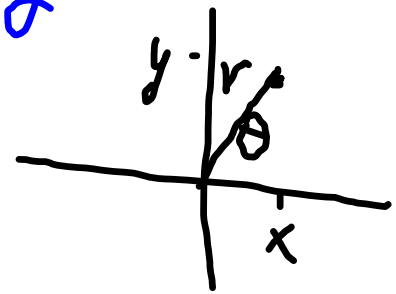
$$\iint f(x,y)\,d\,\text{area}$$

area $(dr)(r\,d\theta)$

$$\iint f\,d\,\text{area}$$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{R} f(r\cos\theta, r\sin\theta)\,dr\,r\,d\theta$$

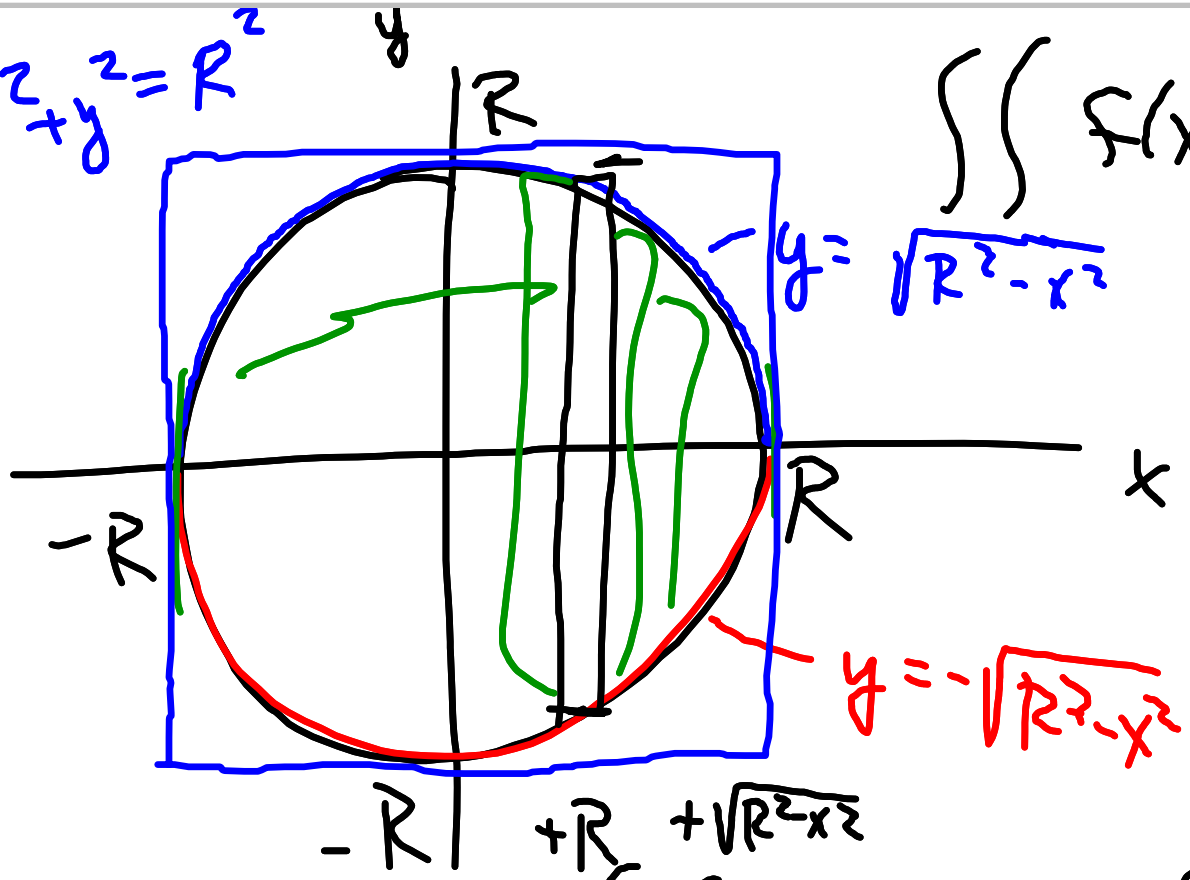
area $= dx\,dy$



$x = r\cos\theta$

$y = r\sin\theta$

$x^2+y^2=R^2$

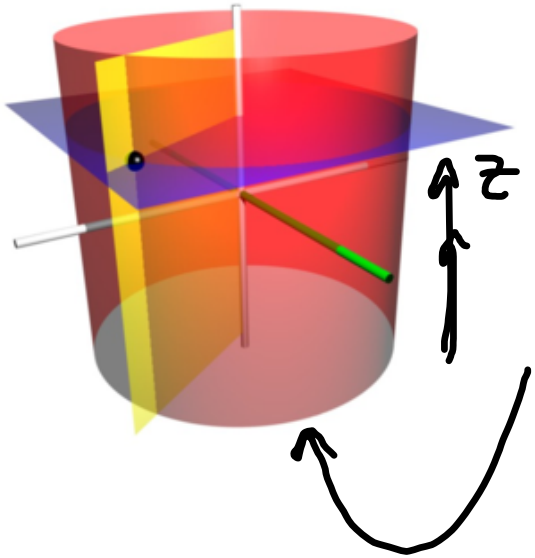


$$\iint f(x,y)\,\underbrace{dx\,dy}_{\text{area}}$$

$$\int_{x=-R}^{+R}\int_{y=-\sqrt{R^2-x^2}}^{+\sqrt{R^2-x^2}} f(x,y)\,dy\,dx$$

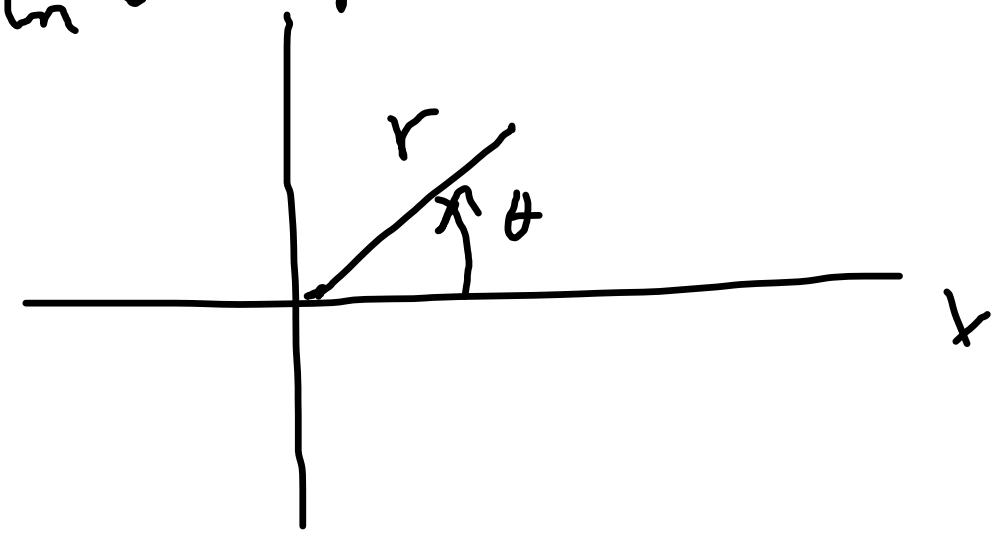
$$=\int_{y=-R}^{R}\int_{x=-R}^{R} f(x,y)\,\phi(x,y)\,dx\,dy$$

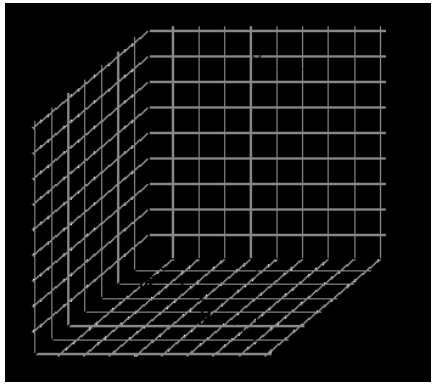
$$\phi(x,y)=\begin{cases}1 & \text{if } x^2+y^2<R^2\\ 0 & \text{if } x^2+y^2>R^2\end{cases}$$



Cylindrical Coords

in $z=0$





(Use r, not ρ, for this lecture.)

$x = r\cos\theta$

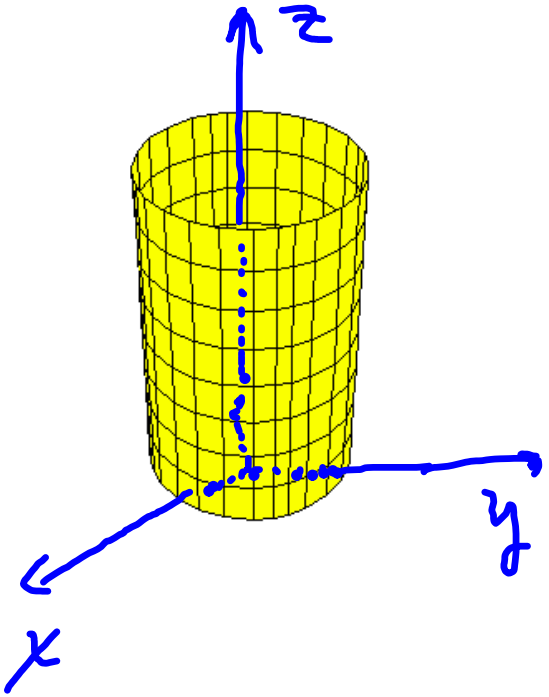
$y = r\sin\theta$

$z = z$

$r^2 = x^2 + y^2$

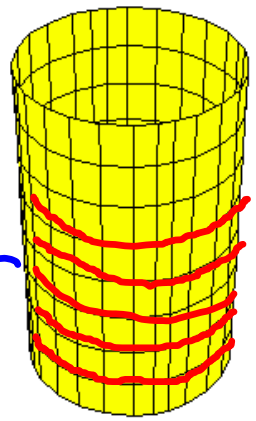
$\tan\theta = y/x$

$z = z$



$r = R$

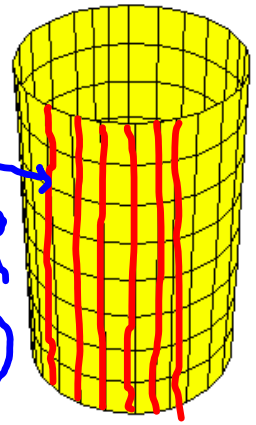
$z = Z$



$0 \le \theta \le 2\pi$

$r = R$

$\theta = \Theta$



$z_1 \le z \le z_2$

$x = r\cos\theta$

$y = r\sin\theta$

$z = z$

Parametrize the circle

$x = R\cos\theta$

$y = R\sin\theta$

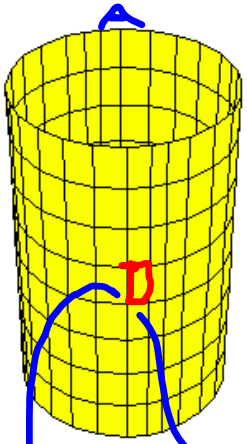
$z = Z$, $0 \le \theta \le 2\pi$

Parametrize the line:

$x = R\cos\Theta$

$y = R\sin\Theta$

$z_1 \le z \le z_2$



$d\,\text{area}$ for $d\theta$ by dz

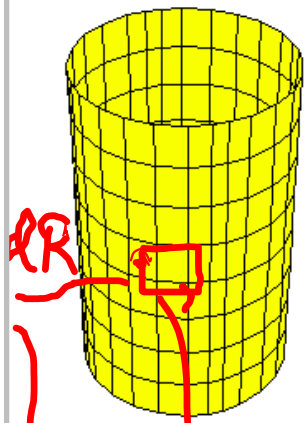
$= (dz)(r\,d\theta)$

$d\theta$ length $= d\theta \cdot r$

dz

$$\iint_{\text{cylinder}} f(x,y,z)\,d\,\text{area}$$

$$= \int_{\theta=0}^{2\pi} \int_{z=z_1}^{z_2} f(R\cos\theta, R\sin\theta, z)\,dz\,R\,d\theta$$

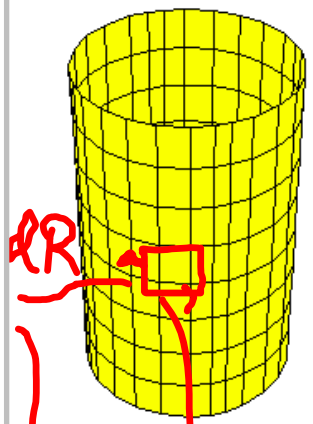


Flux:

$$\iint \vec{F}(x,y,z)\cdot \underbrace{\vec{n}\ \text{darea}}_{\underbrace{d\vec{R}\times d\vec{R}}_{\frac{\partial \vec{R}}{\partial \theta}\times\frac{\partial \vec{R}}{\partial z}\,d\theta\,dz}}$$

$$d\vec{R} = d\vec{R}_\theta = \frac{\partial \vec{R}}{\partial \theta}\,d\theta$$

$$d\vec{R}_z = \frac{\partial \vec{R}}{\partial z}\,dz$$



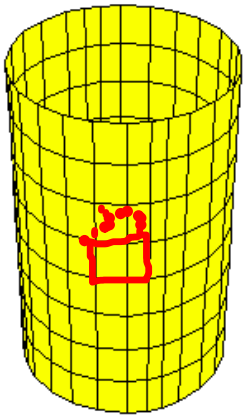
Flux:

$$\iint \vec{F}(x,y,z)\cdot \underbrace{\vec{n}\ d\text{area}}_{\underbrace{d\vec{R}\times d\vec{R}}_{\frac{\partial \vec{R}}{\partial \theta}\times\frac{\partial \vec{R}}{\partial z}\,d\theta\, dz}}$$

$$d\vec{R} = d\vec{R}_\theta = \frac{\partial \vec{R}}{\partial \theta}\, d\theta$$

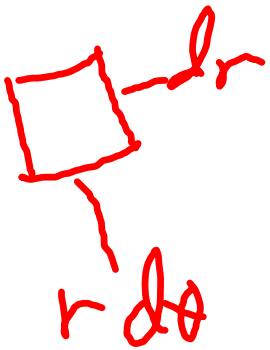
$$d\vec{R}_z = \frac{\partial \vec{R}}{\partial z}\, dz$$

$$\left.\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z \end{aligned}\right\} \quad r = R \quad \vec{R} = (R\cos\theta)\vec{i} + (R\sin\theta)\vec{j} + z\vec{k}$$



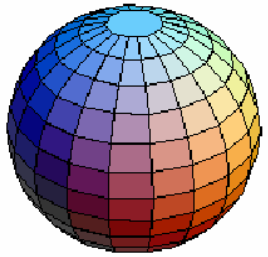
$$\int_{z=z_1}^{z_2}\int_{\theta=0}^{2\pi}\int_{r=0}^{R} f(\underbrace{x}_{r\cos\theta},\underbrace{y}_{r\sin\theta},\underbrace{z}_{z})\,\underbrace{dx\,dy\,dz}_{d\,\text{volume}}$$

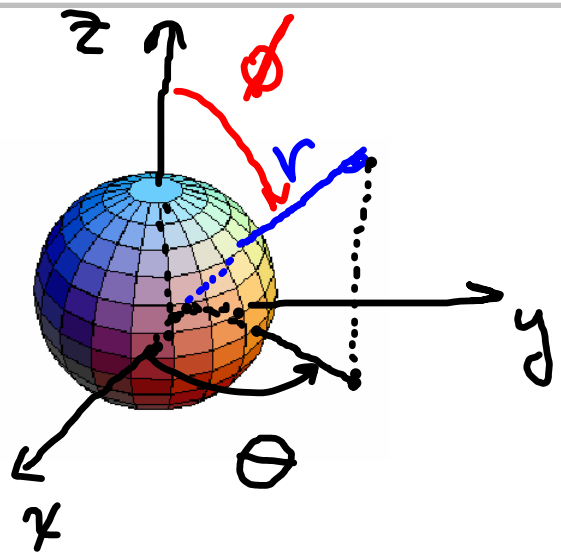
from top $\quad d\,\text{volume} \quad dz \text{ by } d\theta \text{ by } dr$



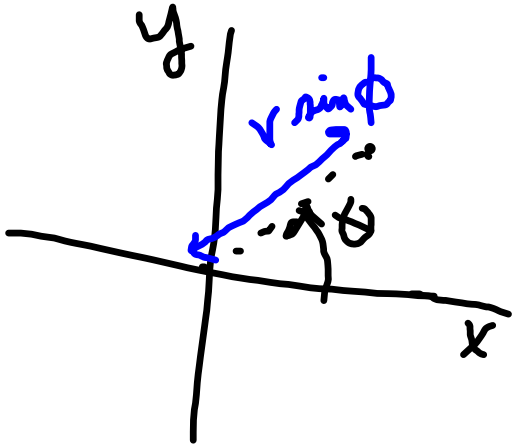
(height = dz)

$$= dr\; r\,d\theta\; dz$$





In x, y plane

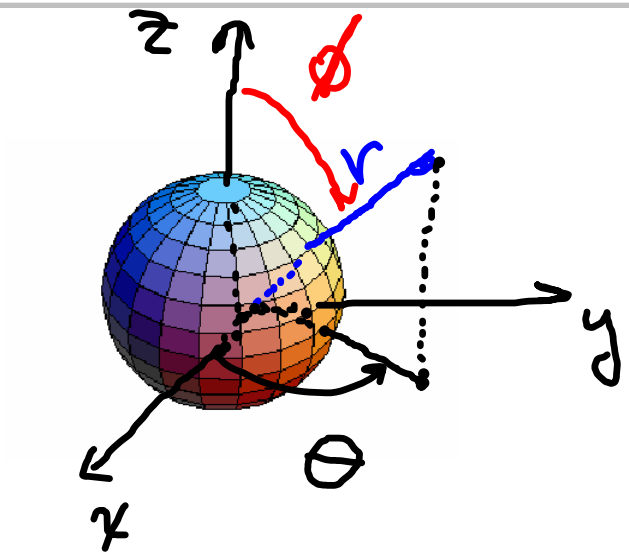


$$x = r \sin\phi \cos\theta$$

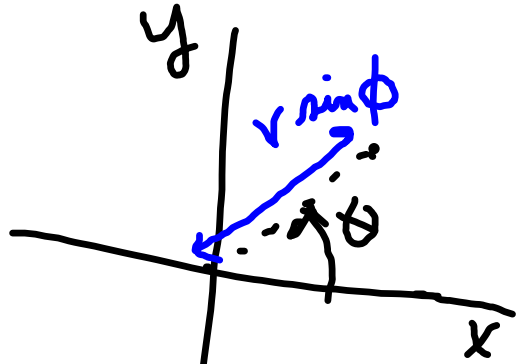
$$y = r \sin\phi \sin\theta$$

$$z = r \cos\phi$$

Flux integral



In x, y plane



$$\iint \vec{F}(x,y,z)\cdot\vec{n}\,d\text{area}$$

$r = R$

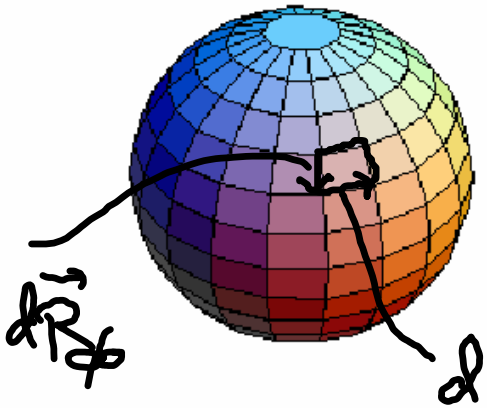
Parametric eqs for sphere

$x = R\sin\phi\cos\theta$

$y = R\sin\phi\sin\theta$

$z = R\cos\phi$

$0 \le \theta \le 2\pi,\ 0 \le \phi \le \pi$



$d\theta$ by $d\phi$

$d\vec{R}_\phi$

$d\vec{R}_\theta$ — length (radius) $d\theta$

$\parallel$

$R\sin\phi$

length $R\,d\phi$

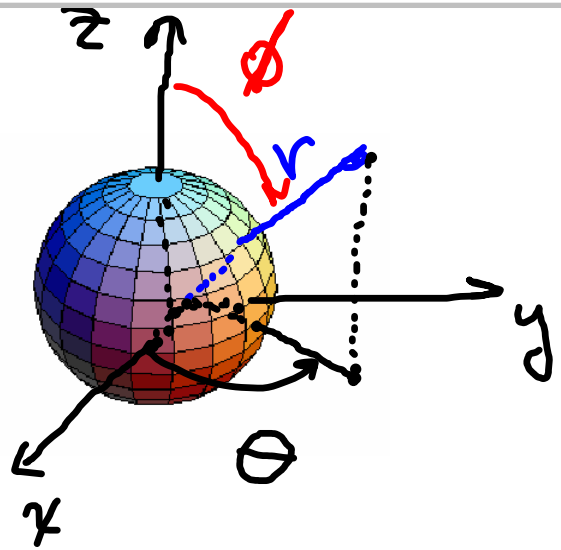
dareea: $R\,d\phi\,R\sin\phi\,d\theta$

~~dareea~~

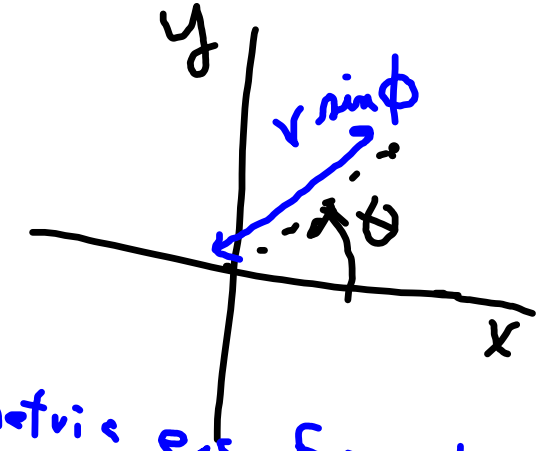
$$d\vec{R}_\phi \times d\vec{R}_\theta = \frac{\partial\vec{R}}{\partial\phi}\times\frac{\partial\vec{R}}{\partial\theta}\,d\phi\,d\theta$$

outward

Flux integral



In x, y plane



$$\iint \vec{F}(x,y,z)\cdot\vec{n}\,d\text{area}$$

$r = R$

$$\frac{\partial\vec{R}}{\partial\phi}\times\frac{\partial\vec{R}}{\partial\theta}\,d\phi\,d\theta$$

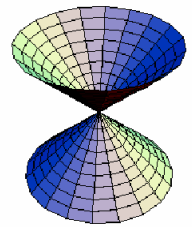
Parametric eqs for sphere

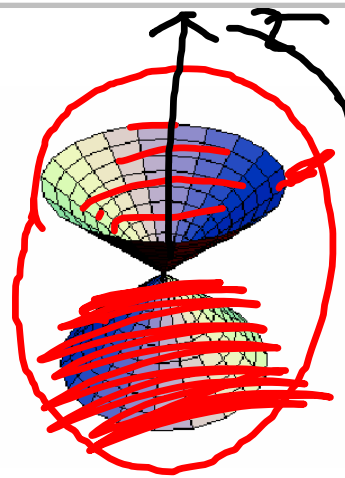
$x = R\sin\phi\cos\theta$

$y = R\sin\phi\sin\theta$

$z = R\cos\phi$

$0 \le \theta \le 2\pi,\ 0 \le \phi \le \pi$



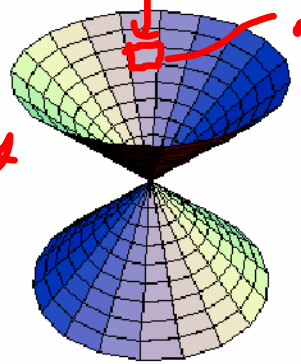


$\phi = \Phi$ constant

length

$= \text{radius}\, d\theta$

$= R \sin\phi \, d\theta$



flux

$$\iint \vec{F}(x,y,z)\cdot\vec{n}\, d\text{area}$$

$$\frac{\partial \vec{R}}{\partial\theta} \times \frac{\partial \vec{R}}{\partial r}\, d\theta\, dr$$

$$\int_{r=0}^{R}\int_{\theta=0}^{2\pi} \ldots \Phi, \theta, r$$

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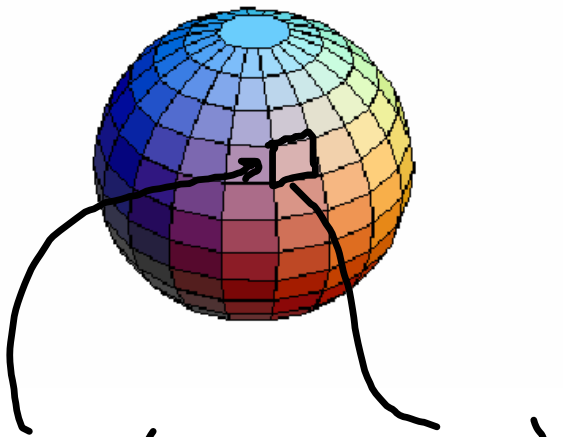
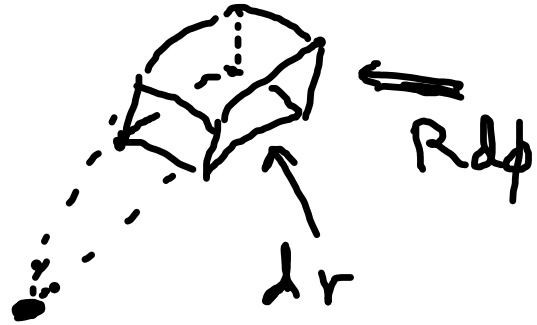


from inside



$R\,d\phi$ $\quad$ (radius) $d\theta$

$\quad\quad\quad