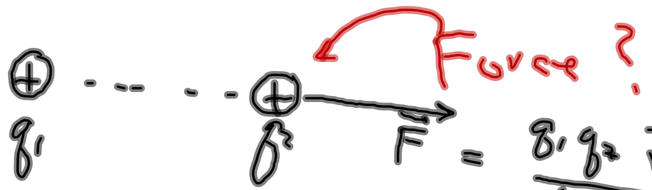
Numerical E/M

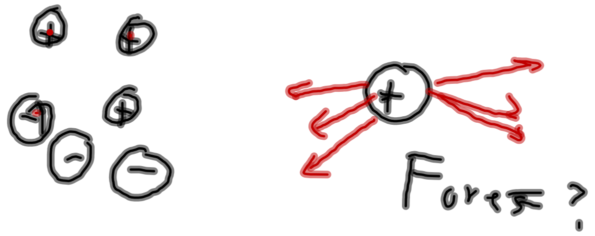
static problems

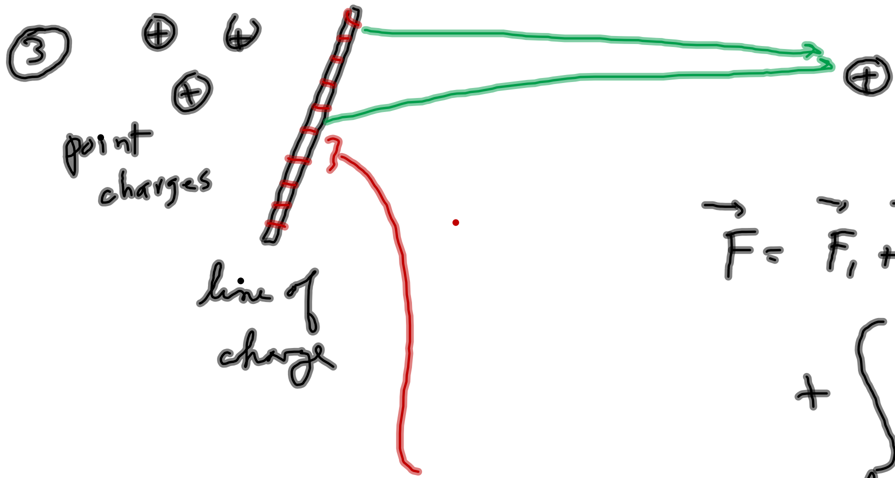
①



$$\vec{F} = \frac{q_1 q_2 \, \vec{n}_{\text{line of sight}}}{(\text{distance})^2} \frac{1}{4\pi\epsilon_0}$$

②





$$\vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 \quad \text{point charge}$$

$$+ \int_{\text{along the line}} \frac{\rho_L \, dL \cdot q_{test}}{\text{distance}^2} \frac{1}{4\pi\varepsilon_0} \vec{r}_{1,02}$$

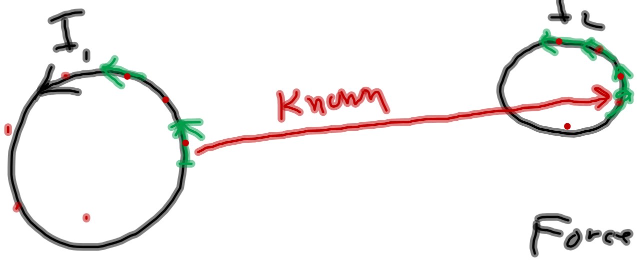
$\Delta l \cdot$ charge density

Coulombs per m

linear charge density

ρ_L

(4) current force
(magnetic)



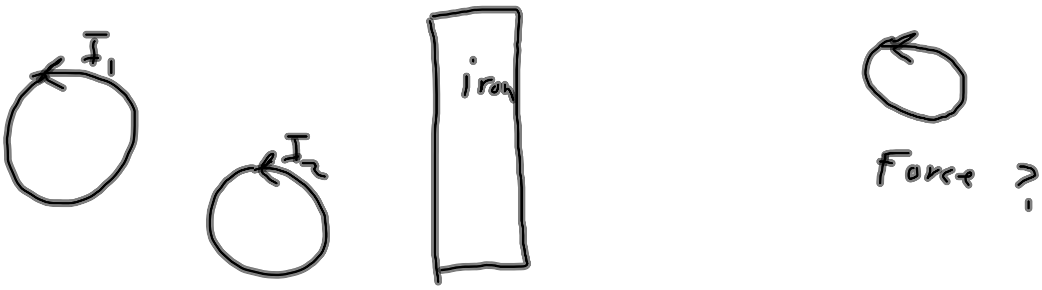
Force on I_2.

$$\int_{\text{test loop}} \int_{\text{source wire}} \overrightarrow{\text{Known}}$$

We will form a model for the conductor.

Use partial differential equation.

6 magnetic

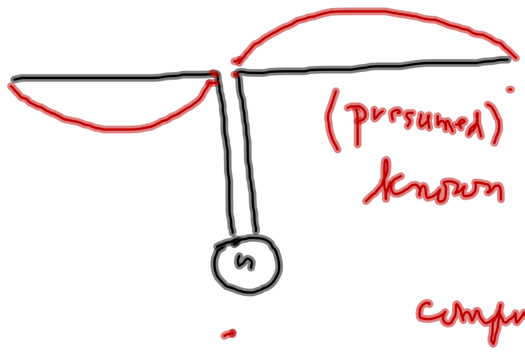


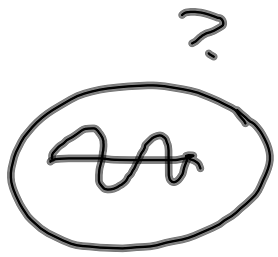
Time-varying

Look at maxwell's eq.

Finite Difference Time Domain

2

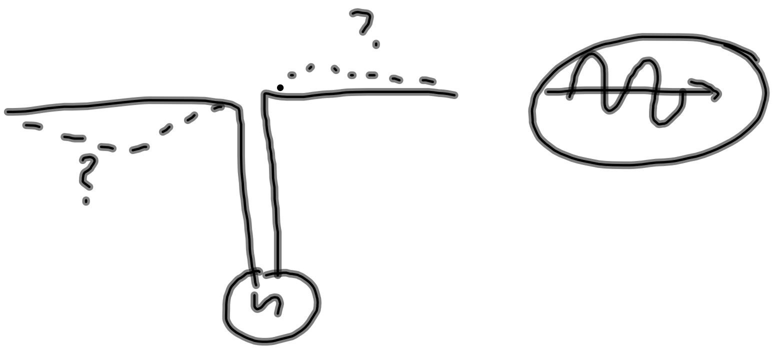




known currents on wire,

compute radiated field.

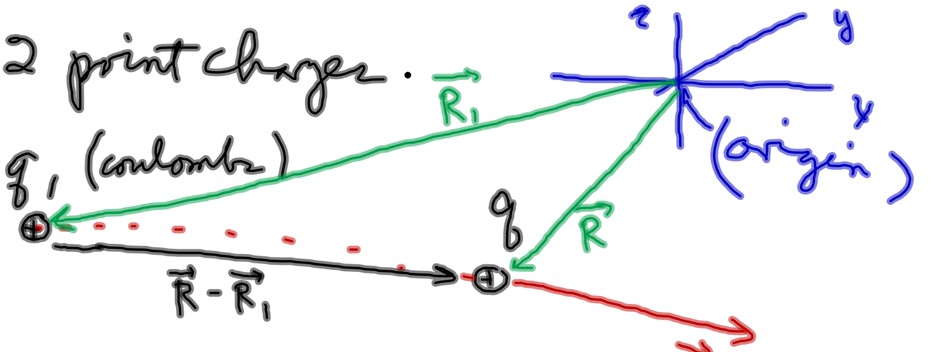
3



You need to Know.

1. Electrophysics - Maxwell's equations
2. Vector Analysis
3. Partial Differential Eqs.
4. Convert PDE's into numerical algorithms.
5. Complex analysis.

2 point charges.



$$\vec{F} = \frac{q_1 q}{4\pi\epsilon_0}\,\frac{\vec{dir}}{(dist)^2}$$

Computer implementation.

Define a coord. system

Label coordinates of charges. $\vec{R}_1 = \{\#, \#, \#\}$

$\vec{R} = \{\#, \#, \#\}$

Write down $\vec{F}$ formula.

$$\vec{dir} = \frac{\vec{R}-\vec{R}_1}{|\vec{R}-\vec{R}_1|} \qquad |\vec{R}-\vec{R}_1| = \sqrt{(x-x_1)^2+(y-y_1)^2+(z-z_1)^2} = dist.$$

$$\vec{F} = \frac{q_1 q}{4\pi\epsilon_0} \frac{(\vec{R} - \vec{R}_1)}{|\vec{R} - \vec{R}_1|^3}$$

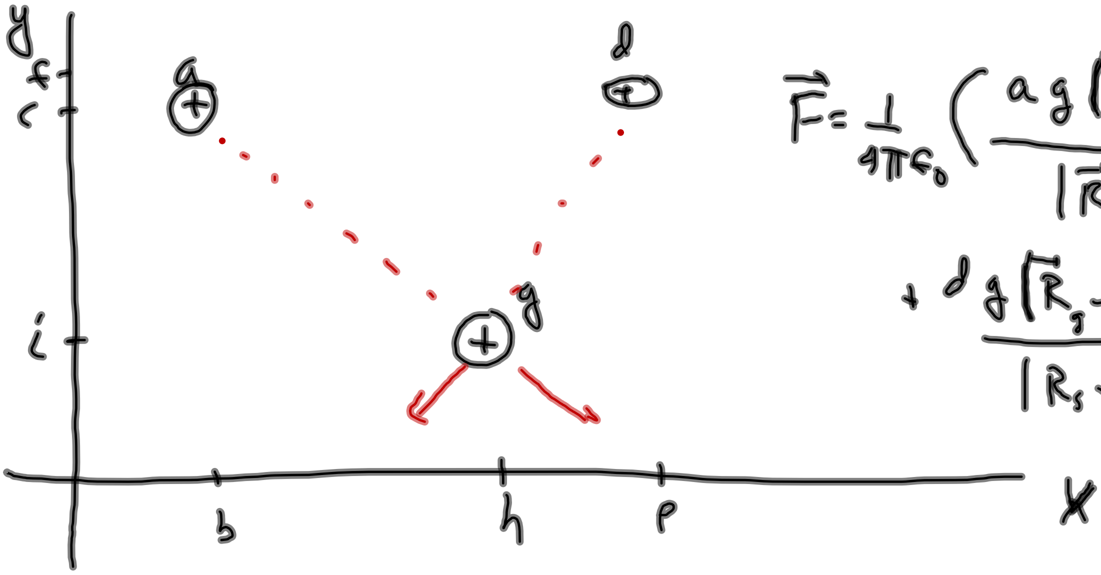
$$R = [x \; y \; z] \qquad R_1 = [X_1 \; Y_1 \; Z_1]$$

MATLAB

$$F = q_1 * q_2 * (R - R_1) * \left(\frac{1}{4\pi\epsilon_0}\right) / \mathrm{dist}^\wedge 3$$

$$\mathrm{dist} = \mathrm{sqrt}\Big(\underbrace{(x-X_1)^\wedge 2 + (y-Y_1)^\wedge 2 + (z-Z_1)^\wedge 2}_{(R-R1)*(R-R_1)'}\Big)$$

for assignment i:



$$\vec{F} = \frac{1}{4\pi\epsilon_0}\left(\frac{a\, g\,(\vec{R}_g - \vec{R}_a)}{|\vec{R}_g - \vec{R}_a|^3} + \frac{d\, g\,(\vec{R}_g - \vec{R}_d)}{|R_g - R_d|^3} \right)$$

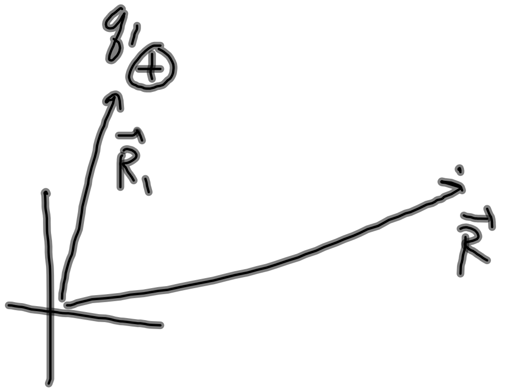
$$\vec{F} = \frac{1}{4\pi\epsilon_0}\left(\frac{a q (\vec{R}_q - \vec{R}_a)}{|\vec{R}_q - \vec{R}_a|^3} + \frac{d q (\vec{R}_q - \vec{R}_d)}{|R_q - R_d|^3} \right) = \boxed{\vec{E}}\, q$$

force per unit test charge.

$$\vec{E} = \frac{q_1}{4\pi\epsilon_0} \frac{\vec{R}-\vec{R}_1}{|\vec{R}-\vec{R}_1|^3}$$

from vector analysis,

this is $\boxed{(-)\vec{\nabla} \frac{q_1}{4\pi\epsilon_0 |\vec{R}-\vec{R}_1|}}$

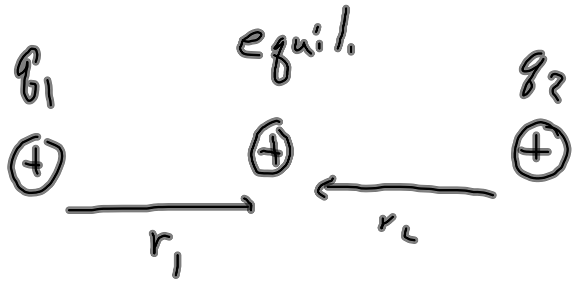


$\vec{\nabla} = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y}$

$\vec{R} = x\vec{i} + y\vec{j} + z\vec{k}$

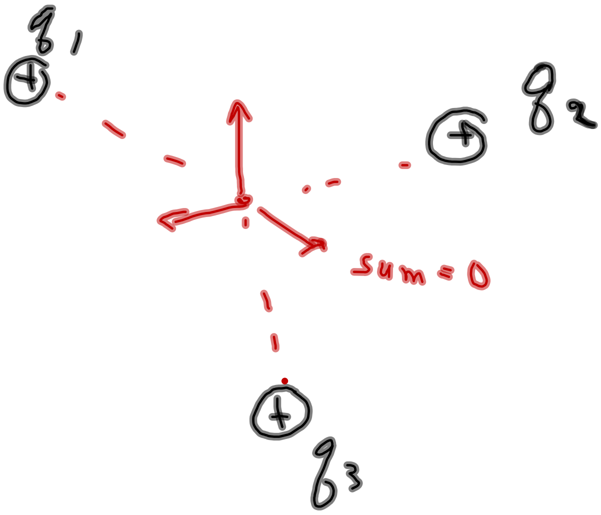
The electric potential at $\vec{R}$ is $\frac{q_1}{4\pi\epsilon_0 |\vec{R}-\vec{R}_1|}$


q_1
test
test
q_2
equilibrium
test
test



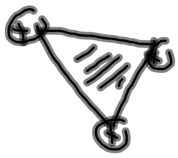
Force to right $= \frac{1}{4\pi\epsilon_0} \frac{q_1}{r_1^2}$

" " left $= \frac{1}{4\pi\epsilon_0} \frac{q_2}{r_2^2}$

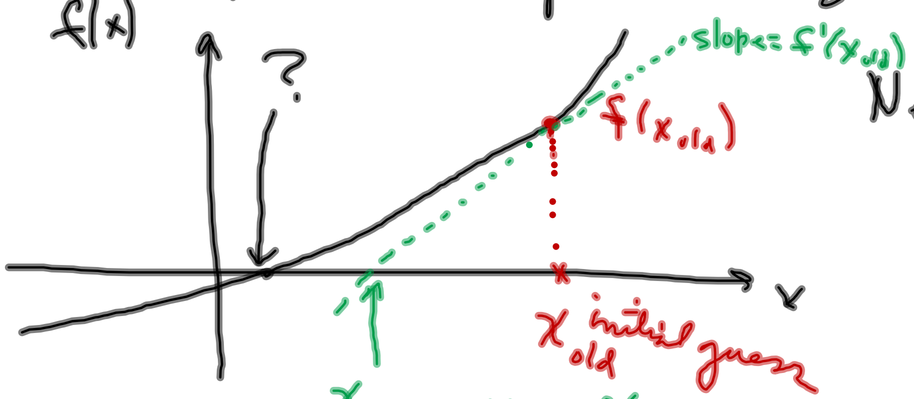


Equilibrium?

certainly inside



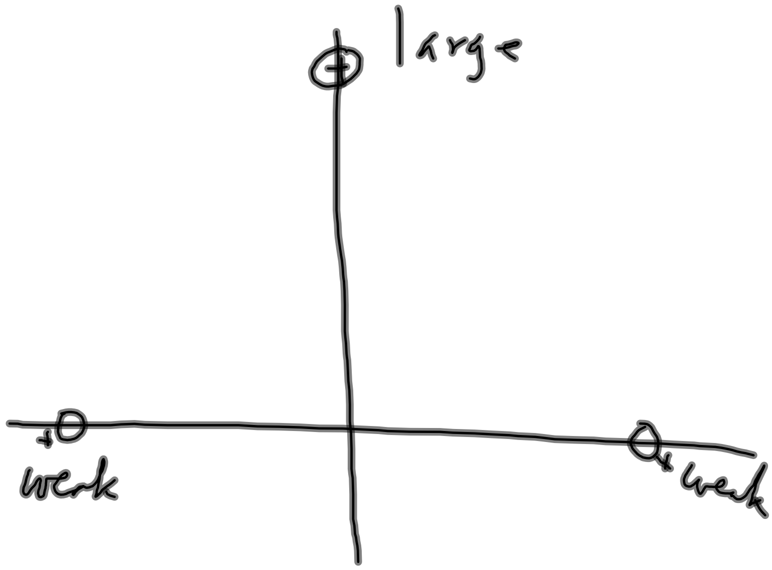
How do find where a function is zero?

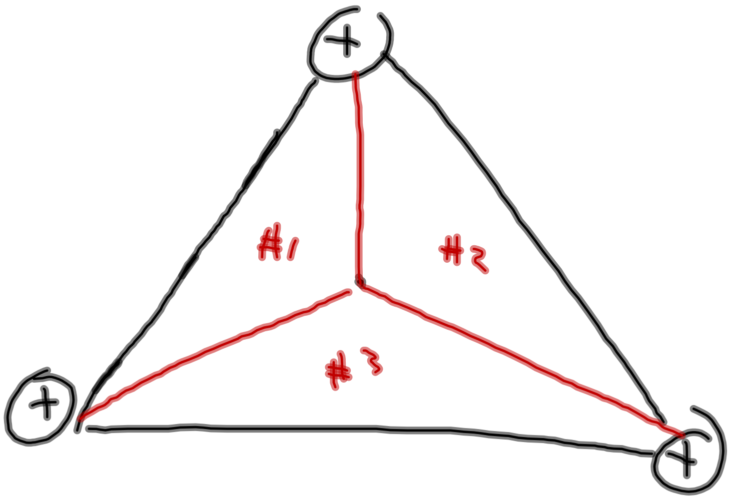


Newton's method.

$$x_{new} = x_{old} - f(x_{old}) / f'(x_{old})$$

$$x_{old} - x_{new} = \frac{\text{opposite}}{\text{slope}} = \frac{f(x_{old})}{f'(x_{old})}$$


large
weak
weak



Expect $\underline{\underline{2}}$ equilibria.

How does Newton find a simultaneous zero of two functions?

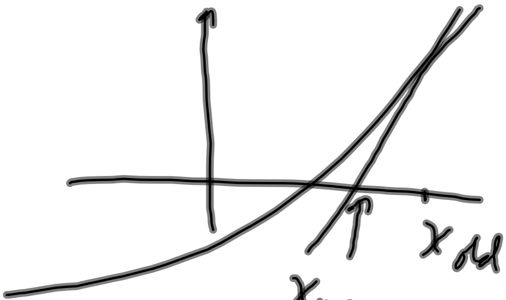
- $f(x,y) = 0$
- $g(x,y) = 0$

$f = 3x + 2y + 7$

$g = 4x - 3y - 2$

$$\begin{bmatrix} 3 & 2 \\ 4 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -7 \\ 2 \end{bmatrix}$$

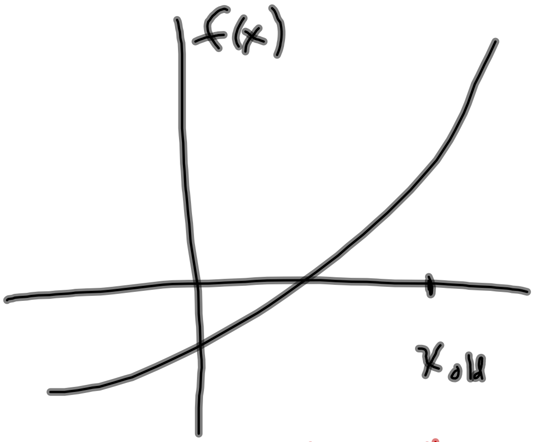
For one function, one variable:



We derived this geometrically.

$$x_{new} = x_{old} - \frac{f(x_{old})}{f'(x_{old})}$$

Now - re-derive it analytically.



Taylor series:

$$\boxed{f(x) = f(x_{old}) + f'(x_{old}) \cdot (x - x_{old})}$$

$$+ f''(x_{old}) \frac{(x - x_{old})^2}{2!}$$

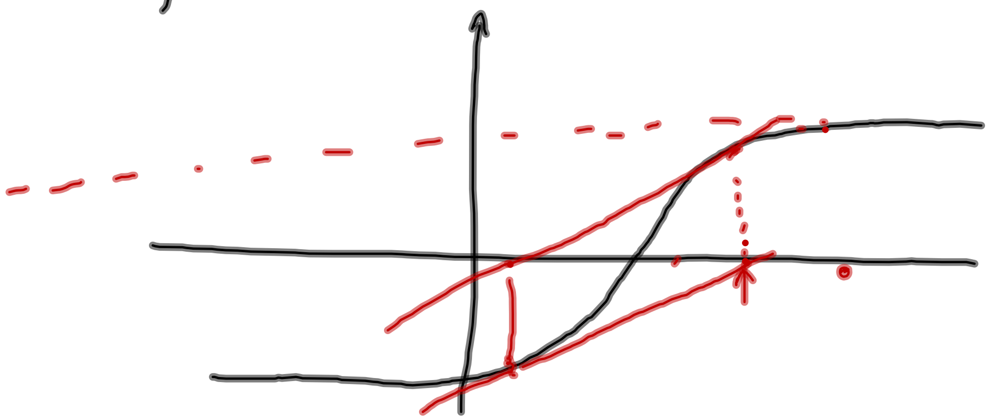
$$+ \ldots$$

We will use $f(x) \approx f(x_{old}) + f'(x_{old})(x - x_{old})$

We want x to make $f(x) = 0$ $\quad 0 = f(x_{old}) + f'(x_{old})(x - x_{old})$

Solve for x: $x = \dfrac{-f(x_{old})}{f'(x_{old})} + x_{old}$

Digression:



Task: we want

$$f(x,y) = 0$$

$$g(x,y) = 0$$

Use Taylor series

$$f(x,y) = f(x_0,y_0) + (x-x_0)\frac{\partial f}{\partial x}(x_0,y_0) + (y-y_0)\frac{\partial f}{\partial y}(x_0,y_0)$$

$$+ \text{(ignore)}$$

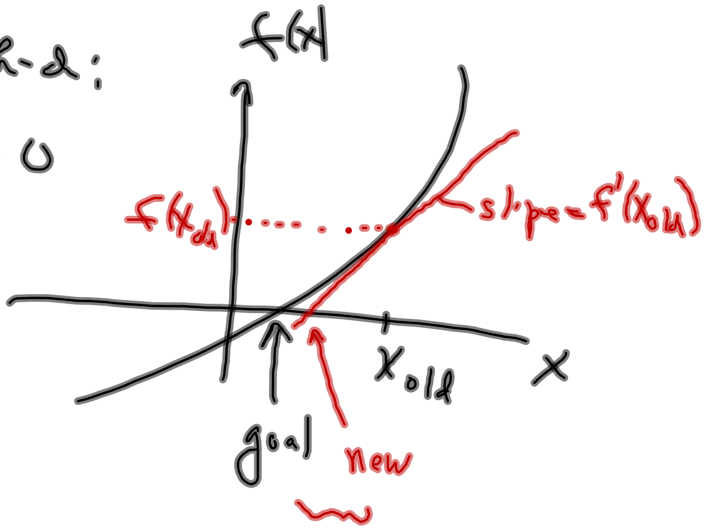
$$g \qquad =$$

Newton-Raphson Method:

to solve $f(x)=0$

iterate

$$x_{new} = x_{old} - \frac{f(x_{old})}{f'(x_{old})}$$



$$x_{new} - x_{old} = -\frac{f(x_0)}{f'(x_0)}$$

Taylor series:

$$f(x) = f(x_0) + f'(x_0)\,[x - x_0]$$

want x so that $f(x) = 0$:

$$0 = f(x_0) + f'(x_0)\,[x - x_0]$$

solve for x

$$x = x_{old} - \frac{f(x_0)}{f'(x_0)}$$

2-dimensions.

Simultaneous sol'n $f(x,y)=0$

$g(x,y)=0$

Taylor series:

$$f(x,y) = f(x_0,y_0) + \left.\frac{\partial f}{\partial x}\right|_{x_0,y_0}(x-x_0) + \left.\frac{\partial f}{\partial y}\right|_{x_0,y_0}(y-y_0)$$

Same for $g(x,y)$:

Goal: want (x,y) which make $f(x,y)=0$, $g(x,y)=0$

$$0 = f(x_0,y_0) + \frac{\partial f}{\partial x}(x-x_0) + \frac{\partial f}{\partial y}(y-y_0)$$

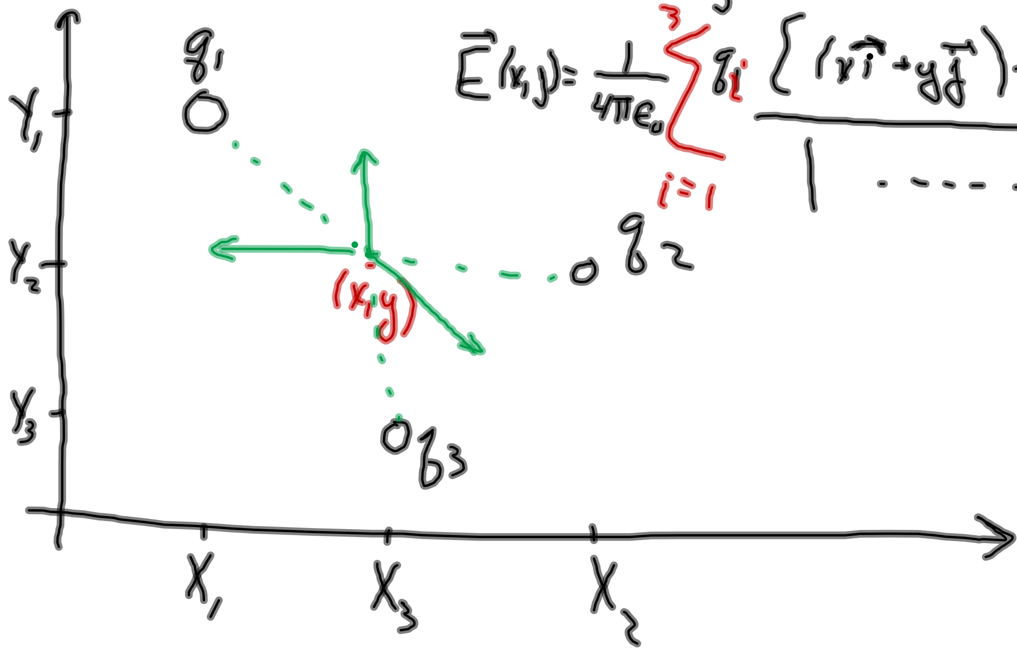
$$0 = g \quad - - - - - -$$

(I think)

$$\begin{bmatrix} x \\ y \end{bmatrix}_{new} = \begin{bmatrix} x \\ y \end{bmatrix}_{old} - \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial x} \end{bmatrix}_{old}^{-1} \begin{bmatrix} f \\ g \end{bmatrix}_{old}$$

For the Coulomb assignment

$$\vec{E}(x,y) = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{3} q_i \frac{\left[(x\vec{i} + y\vec{j}) - (x_i\vec{i} + y_i\vec{j}) \right]}{\left| \cdots\cdots \right|^3}$$

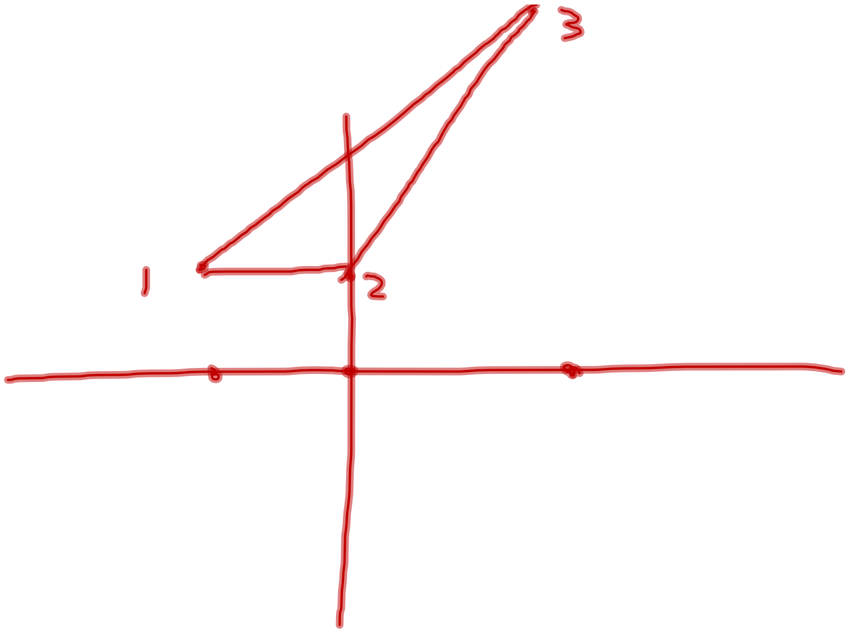


In the context of $f(x,y)=0$, $g(x,y)=0$,

$$f(x,y) = E_1(x,y) = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{3}\frac{(x-X_i)\,q_i}{\sqrt{(x-X_i)^2+(y-Y_i)^2}^{\,3}}$$

$$g(\text{—}) = \quad - \quad \cdots\cdots \; y-Y_i \;\cdots$$

$$\frac{\partial f}{\partial x} =$$


3
1
2

Reformulate in terms of potential.

Zero-force means flat potential (max or min)

Math $\Rightarrow$ there are no minima in an electrostatic field.

(There are saddle points.)

What is the potential?

$$\vec{E}(\vec{R}) = \frac{q_i}{4\pi\epsilon_0} \frac{|\vec{R} - \vec{R}_i|}{|\vec{R} - \vec{R}_i|^3} \quad \text{summed over } i.$$

$$\vec{E}(\vec{R}) = -\nabla\phi$$

ANSWER: $\dfrac{q_i}{4\pi\epsilon_0 |\vec{R} - \vec{R}_i|}$

alternative soln :

$$\phi(x,y) = \sum_{i=1}^{3} \frac{q_i}{4\pi\epsilon_0} \frac{1}{|\ \text{—}\ |}$$

$$\sqrt{(x-x_i)^2+(y-y_i)^2}$$

Gradient:

start (x, y).

"step size"

$$\begin{bmatrix} x_{new} \\ y_{new} \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}_{old} + \begin{bmatrix} \frac{\partial \phi}{\partial x} \\ \frac{\partial \phi}{\partial y} \end{bmatrix}_{old} s$$

The electrostatic equilibrium occurs at a point where the potential is maximum.

How about searching for maximum potential, instead of searching for zero field?

1. Easier to calculate.
2. If you're at the max, the gradient is zero.
3. If you're not at the max, the gradient is the best direction to go, to increase the potential.

SO...,

the method of steepest ascent:

$$\begin{bmatrix} x \\ y \end{bmatrix}_{new} = \begin{bmatrix} x \\ y \end{bmatrix}_0 + \left[\vec{\nabla}\phi\right] s$$

step size? How far?

If you happen to know the value of the potential at its maximum, you could use the gradient (rate of change) to estimate the desired step size.

Possible strategy:

at $\begin{bmatrix} x \\ y \end{bmatrix}_{old}$, compute $\phi_{potential}$, compute $\nabla\phi$,

compute $\underbrace{\text{derivatives of the gradient of the potential}}_{\text{"}\nabla\nabla\phi\text{"}}$

fit a parabola in the gradient direction,

use $\left[Ax^2+Bx+C\right]_{min}$ at $x=-\frac{B}{2A}$ to estimate the step size

NO - This is exactly the same as Newton-Raphson, operating on the E-field directly.

Potentials: $\vec{F}_{orce} = -\nabla \Phi_{scalar}$, ϕ is the "potential energy"

$\vec{E} = -\nabla V$ V is the potential (volts).

Vector analysis theory:

If $\vec{E} = -\nabla V$, $\vec{\nabla} \times \vec{E} = (-) \vec{\nabla} \times \vec{\nabla} V \equiv \vec{0}$

$\vec{E} = -\nabla V = -\vec{i} \frac{\partial \phi}{\partial x} - \vec{j} \frac{\partial \phi}{\partial y}$ ——

$$\vec{\nabla} \times \vec{\nabla} V = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_1 & E_2 & E_3 \end{vmatrix} = - \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & - \\ \frac{\partial V}{\partial x} & \frac{\partial V}{\partial y} & - \end{vmatrix} =$$

$$= \left(\frac{\partial}{\partial x} \frac{\partial V}{\partial y} - \frac{\partial}{\partial y} \frac{\partial V}{\partial x} \right) \vec{K} + \text{others}$$

$$= 0$$

Remember Maxwell's eq.

$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ in electrostatics, $\frac{\partial}{\partial t} = 0$

so $\nabla \times \vec{E} = 0$ for electrostatics

So, is $\vec{E} = -\nabla V$? Vector analysis says "Yes."

BIG DEAL : $\vec{E} = -\nabla(\text{something})$

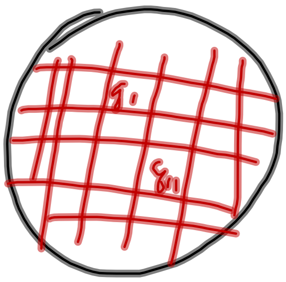
Recall $$\vec{E}(\vec{R}) = \sum_i \frac{q_i}{4\pi\epsilon_0} \frac{(\vec{R}-\vec{R}_i)}{|\vec{R}-\vec{R}_i|^3} = -\sum_i \nabla \frac{q_i}{|R-R_i|}\frac{1}{4\pi\epsilon_0}$$

$$\text{Potential} = \sum_i \frac{q_i}{4\pi\epsilon_0 |\vec{R}-\vec{R}_i|}$$

q_1 q_2 q_3 $\cdots$ q_{1000}

$$\vec{E} = \sum_{i=1}^{1000} \frac{q_i \, (\vec{R} - \vec{R}_i)}{4\pi\epsilon_0 \, |\vec{R} - \vec{R}_i|^3}$$

Continuum volume of charge



$dx\,dy\,dz$

$\rho\, dV_{volume}$

$$\vec{E} = \iiint \frac{\cancel{q_i}\,(\vec{R}-\vec{R}_i)}{4\pi\epsilon_0 \left|\vec{R}-\cancel{\vec{R}_i}\right|^3}$$

$\vec{R}'$

q_{11} = charge density · volume of cell 11.
(coulomb per m^3)

$$\vec{E}(\vec{R}) = \iiint_{volume} \frac{\rho(\vec{R}')\; dx'\, dy'\, dz'}{4\pi\epsilon_0 \left|\vec{R}-\vec{R}'\right|^3}$$

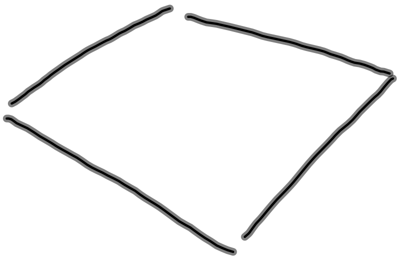


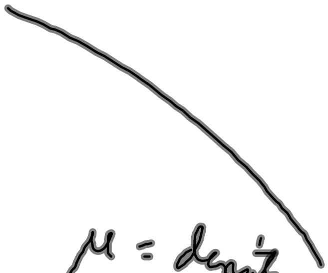
plate of charge

$$\vec{E} = \iint \frac{\sigma(\vec{R}')\,(\vec{R}-\vec{R}')\,d_{area}}{4\pi\epsilon_0\,|\vec{R}-\vec{R}'|^3}$$

σ = charge per unit area

(~~cola~~ C/m^2)

line source

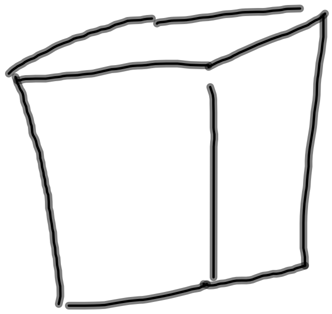


μ = density

= C/m

$$\vec{E} = \int \frac{\mu(\vec{R}')\,(\vec{R}-\vec{R}')\,dl}{4\pi\epsilon_0\,|\vec{R}-\vec{R}'|^3}$$

q_1 q_2 q_3



solid conductor.

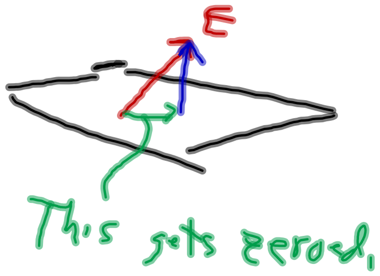
$$\vec{E} = \iiint \frac{\rho \, dV \cdots}{\cdots}$$

$$V = \iiint \frac{\rho \, dV}{|\;|\, 4\pi\varepsilon_0}$$

Model of an ideal solid conductor:

no $\vec{E}$-field inside (static), because there are always electrons available to shield it.

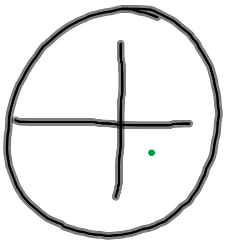
model of ideal plate conductor.

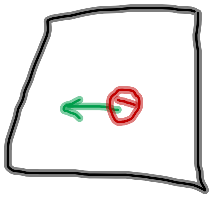


electrons move to cancel $\vec{E}$ field parallel to the plate.

Elaboration

1^{st} moment



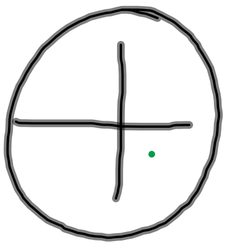


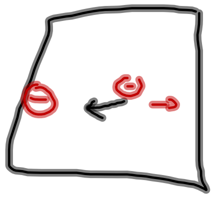
solid

conductor

Elaboration

2nd ~~1st~~ moment



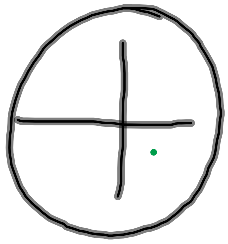


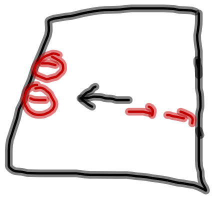
solid

conductor

Elaboration

3rd ~~2nd~~ ~~1st~~ moment

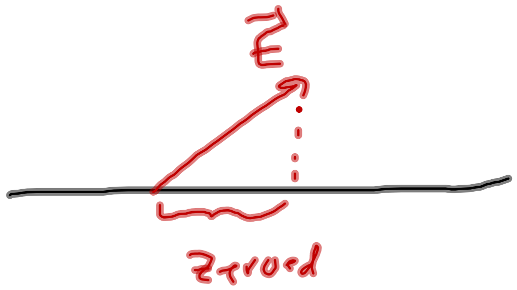




no interior $\vec{E}$ field.

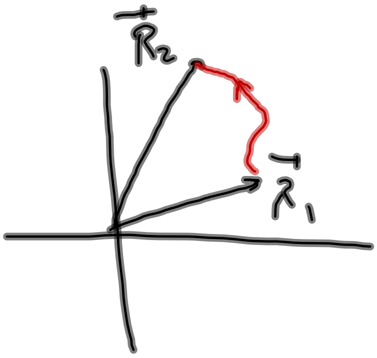
solid conductor

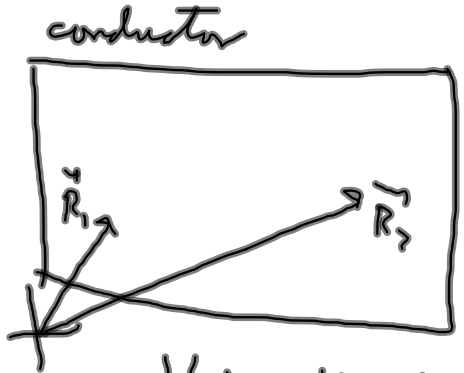
Conducting wire



Theory $\vec{E} = -\nabla V$

$$V(\vec{R}_1) - V(\vec{R}_2) = -\int \vec{E} \cdot d\vec{R}$$

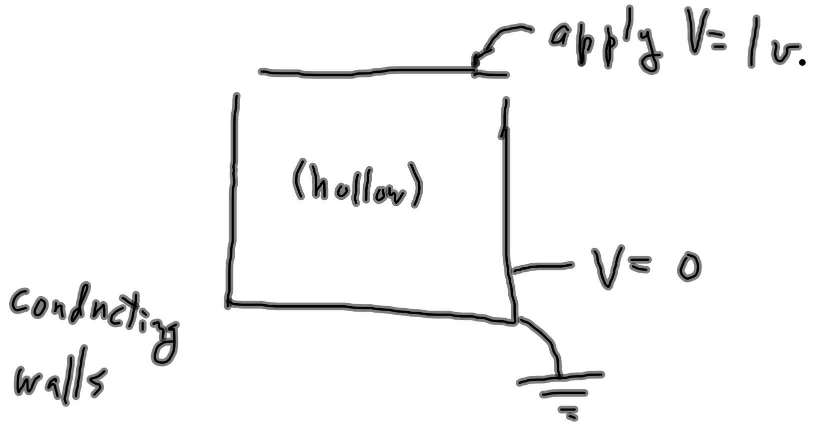




$$\vec{V}(\vec{R}_1) - V(\vec{R}_1) = \int \vec{0} \cdot d\vec{R}$$

V is uniform inside a conductor

Important conclusion: in electrostatics, conductors are equipotentials.



What is the electric field inside?
" " " potential "

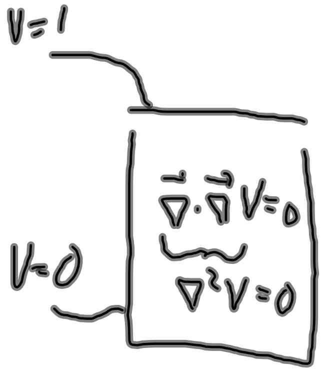
Don't know the charge distribution:

need another formulation.

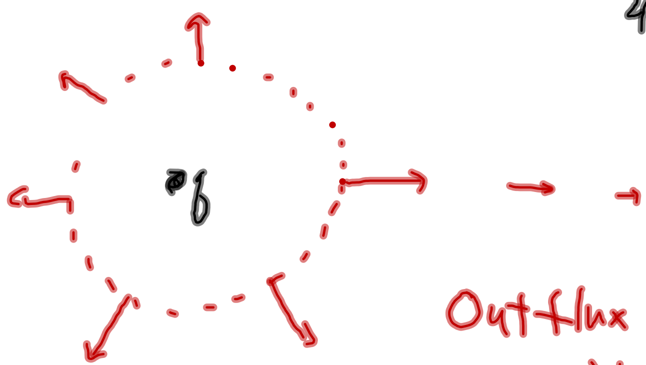
Maxwell $\quad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = 0 \quad \left(\vec{E} = -\nabla V\right)$

$$\nabla \cdot \vec{E} = \frac{1}{\epsilon_0}\rho = 0 \text{ inside box}$$

charge volume density



Coulomb's law. $\vec{E}(\vec{R}) = \dfrac{1}{4\pi\epsilon_0} \dfrac{q\,(\vec{R}-\vec{R}')}{|\vec{R}-\vec{R}'|^3}$

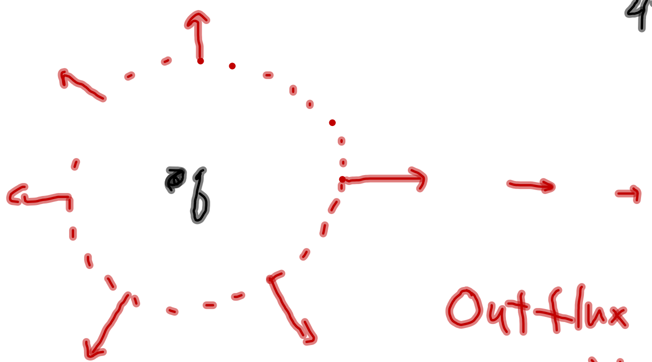


Outflux of $\vec{E}$ through a sphere

$= |\vec{E}| \cdot$ area of sphere

$= \dfrac{q}{4\pi\epsilon_0 \,\text{radius}^2} \underbrace{4\pi\,\text{radius}^2}_{} = q/\epsilon_0$

Coulomb's law. $\vec{E}(\vec{R}) = \frac{1}{4\pi\epsilon_0} \frac{q\,(\vec{R}-\vec{R}')}{|\vec{R}-\vec{R}'|^3}$



Outflux of $\vec{E}$ through a sphere

$= |\vec{E}| \cdot$ area of sphere

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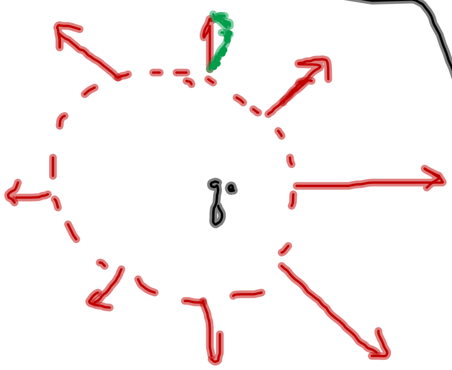
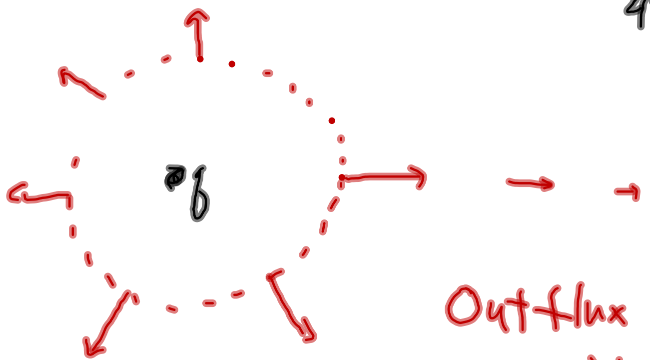


Outflux of $\vec{E}$: (strength of $\vec{E}$ varies)
($\vec{E}$ is not normal to sphere)

Math Fact: same flux.

$$\frac{q}{\epsilon_0} = \iint_{\text{sphere}} E_n \, dA$$

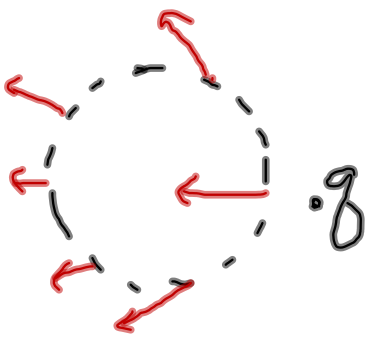
Coulomb's law. $\vec{E}(\vec{R}) = \frac{1}{4\pi\epsilon_0} \frac{q\,(\vec{R}-\vec{R}')}{|\vec{R}-\vec{R}'|^3}$



Outflux of $\vec{E}$ through a sphere

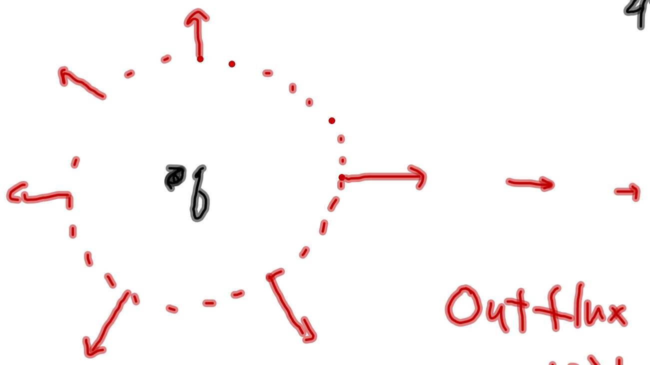
$= |\vec{E}| \cdot$ area of sphere

$= \frac{q}{4\pi\epsilon_0 \,\text{radius}^2} \; 4\pi \,\text{radius}^2 = q/\epsilon_0$



Math fact: outflux is zero.

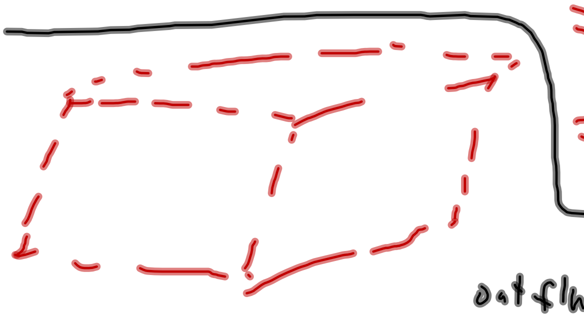
Coulomb's law. $\vec{E}(\vec{R}) = \frac{1}{4\pi\epsilon_0} \frac{q\,(\vec{R}-\vec{R}')}{|\vec{R}-\vec{R}'|^3}$



Outflux of $\vec{E}$ through a sphere

$= |\vec{E}| \cdot$ area of sphere

$= \frac{q}{4\pi\epsilon_0 \text{ radius}^2} \underbrace{4\pi \text{ radius}^2} = q/\epsilon_0$



$$\text{outflux} = \iint_{\text{surface area}} E_n \, dA = \begin{cases} q/\epsilon_0 & \text{if inside} \\ 0 & \text{if outside} \end{cases}$$

distribution of charge

$$\text{outflow} = \iint E_n \, dA = \frac{\text{total charge enclosed}}{\epsilon_0}$$

Maxwell's first law.

If the charge is distributed over volume with a density ρ_v (C/m^3),

$$\iint_{\text{surface}} \epsilon_0 E_n \, dA = \iiint_{\substack{\text{volume}\\ \text{enclosed}}} \rho_v \, d\,\text{Volume}$$

Divide by volume

shrink volume to zero,

$$\underbrace{\epsilon_0\left[\frac{\partial}{\partial x}E_1 + \frac{\partial}{\partial y}E_2 + \frac{\partial}{\partial z}E_3\right]}_{\vec{\nabla}\cdot\vec{E}} = \rho_v \Rightarrow \nabla \cdot \epsilon_0 \vec{E} = \rho_v$$

M's 1st Diff eq.

$$\left(\vec{\nabla} = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)$$

What if you have discrete charges q_i,

surface charge density ρ_s ?,

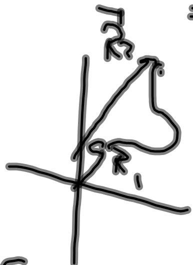
line " " ρ_l

Next topic:

$$\vec{E} = \frac{q\,(\vec{R} - \vec{R}')}{4\pi\epsilon_0 |\,—\,|^3} = -\vec{\nabla}\,\frac{q}{4\pi\epsilon_0 |\,—\,|}$$

Math Fact

If $\vec{F} = \vec{\nabla}\phi$ then $\displaystyle\int_{\vec{R}_1}^{\vec{R}_2} \vec{F}\cdot d\vec{R} = \phi(\vec{R}_2) - \phi(\vec{R}_1)$



any curve

If $\vec{F}$ is force, $\int \vec{F}\cdot d\vec{R}$ is work, $\phi(\vec{R})$ potential energy.

" , fights a field, $-\phi(\vec{R})$ is pot. energy.

So ..... $\frac{q}{4\pi\epsilon_0|-|}$ is potential ~~energy per unit charge~~

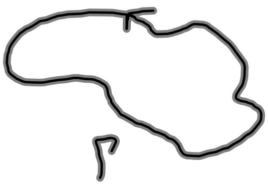
$$\vec{E}(\vec{R}) = \sum_i \frac{q_i(\vec{R}-\vec{R}_i')}{4\pi\epsilon_0|-|^3} + \iiint \frac{\rho_v\, dV(\quad)'}{4\pi\epsilon_0|-|^3}$$

$$+ \iint \frac{\rho_s\, dA(\quad)'}{4\pi\epsilon|-|^3} + \int \frac{\rho_l\, dR(\quad)}{4\pi\epsilon|-|^3}$$

pot.

$$V = \sum \frac{q_i}{4\pi\epsilon_0|\ \ |'} + \iiint \frac{\rho_v\, dV}{4\pi\epsilon_0|\ \ |'} + \ldots \quad ..;$$

So: $$\oint_\Gamma \vec{E}\cdot d\vec{r} = -\left[V_{total}(end) - V(start)\right]$$



$= 0$ for ~~ ~~ closed curve.

M's second law for electrostatics:

$$\oint_{\text{any }\Gamma} \vec{E}\cdot d\vec{r} = 0$$

Math: divide by area enclosed by the curl, shrink the curve to a point.

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_1 & E_2 & E_3 \end{vmatrix}$$ – component normal to ~~the~~ area $= 0$

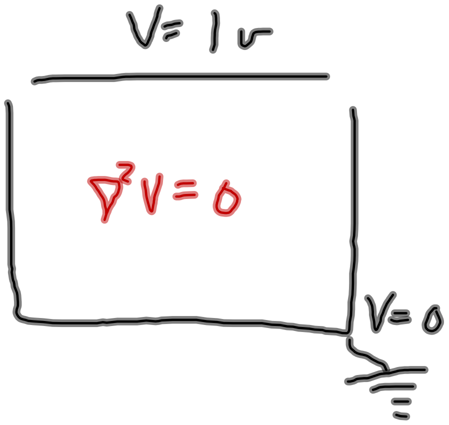
Math: $$\vec{\nabla}\times\vec{E} = 0$$ M's 2nd eq. for electrostatics

For electrostatics:

Laplacian

div $\vec{\nabla}\cdot(\epsilon_0 \vec{E}) = \rho$ $\qquad \dfrac{\rho}{\epsilon_0} = \nabla\cdot\vec{E} = -\nabla\cdot\nabla V = -\nabla^2 V$

curl $\vec{\nabla}\times\vec{E} = 0$



Math Fact: this is enough to determine V!

Knowing V we can $-\vec{\nabla} V = \vec{E}$!

Knowing $\vec{E}$ we can $\vec{\nabla} \cdot \epsilon_0 \vec{E} = \rho$!

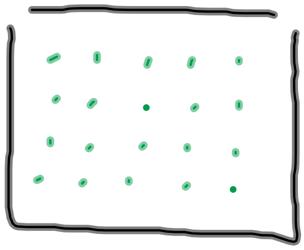
$V_{inside} = ?$

Not only is V determined, we can compute it.

We need algorithms

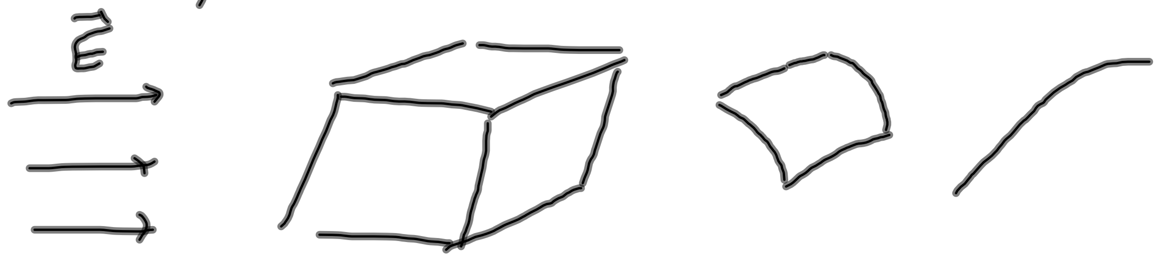
① to express $\nabla^2 V = 0$

② to " $\vec{E} = -\vec{\nabla}\phi$.



Last time,

(solid box) (plate) (wire)



conductor

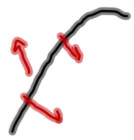
$\vec{E}_{inside} = 0$ $\vec{E}_{tangential} = 0$

Where is the charge?

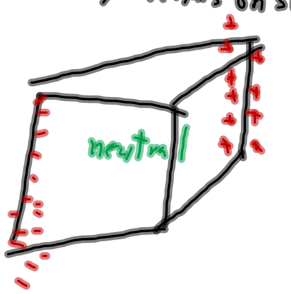
Inside: $\rho_v = \vec{\nabla} \cdot \epsilon_0 \vec{E}$
$= \vec{\nabla} \cdot \epsilon_0 \vec{0}$
$= 0$

charge resides on surface (only)

$\rho_s \neq 0$



$\rho_L \neq 0$



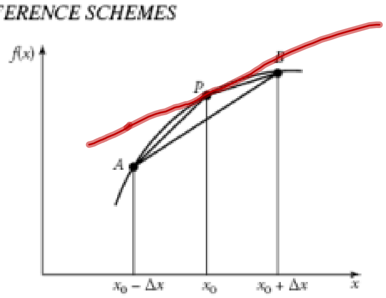


Figure 3.2

Estimates for the derivative of $f(x)$ at P using forward, backward, and central differences.

$$\boxed{f'(x_o) \simeq \frac{f(x_o + \Delta x) - f(x_o)}{\Delta x}} \tag{3.1}$$

or the slope of the arc AP, yielding the *backward-difference* formula,

$$\boxed{f'(x_o) \simeq \frac{f(x_o) - f(x_o - \Delta x)}{\Delta x}} \tag{3.2}$$

or the slope of the arc AB, resulting in the *central-difference* formula, Forward Differences

Backward "

Centered "

Why are CD's much better than FD's & BD's?

Taylor series: $f(x) = f(x_0) + f'(x_0)(x-x_0) + f''(x_0)\frac{(x-x_0)^2}{2!} + f'''(x_0)\frac{(x-x_0)^3}{3!} + \ldots$

FD: $x = x_0 + \Delta x$ $\quad f(x_0+\Delta x) = f(x_0) + \underbrace{f'(x_0)}_{\text{solve}}\Delta x + f''\frac{\Delta x^2}{2} + f'''\frac{\Delta x^3}{6}$

$$f'(x_0) = \frac{f(x_0+\Delta x) - f(x_0)}{\Delta x} + \text{error} \qquad \text{error} = -\frac{f''\Delta x^2}{2\Delta x} - \frac{f'''\Delta x^3}{\Delta x\, 6}$$

$-\frac{f''(x_0)}{2}\Delta x$

BD: $x = x_0 - \Delta x$ $\quad f(x_0-\Delta x) = f(x_0) - f'(x_0)\Delta x + \frac{f''\Delta x^2}{2} - \frac{f'''\Delta x^3}{6}$

$$f'(x_0) = \frac{f(x_0-\Delta x) - f(x_0)}{-\Delta x} + \text{error}$$

$\text{error} = +\frac{f''\Delta x}{2}$

CD: $$\frac{f(x_0+\Delta x) - f(x_0-\Delta x)}{2\Delta x} = f'(x_0)\frac{2\Delta x}{2\Delta x} + \underbrace{\frac{2f'''(x_0)\Delta x^3}{6\cdot 2\Delta x}}_{O(\Delta x^2)}$$

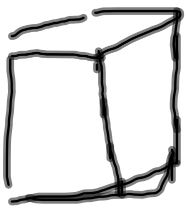
$\vec{E} = -\nabla V(\vec{R}) \qquad \nabla \cdot (\epsilon_0 \vec{E}) = \rho_{vol}$

$\nabla^2 V = -\rho/\epsilon_0 \qquad$ outflux of $\epsilon_0 E$ = charge enclosed

Task: to solve this eq.

Take ∇V to get $\vec{E}$, take $\nabla^2 V$ or $\nabla \cdot E$ or outflux to get charges.

Solid conductor:



$\vec{E} = 0$ inside

$\rho_{vol} = \nabla \cdot (\vec{0}) = 0$

Sheet conductor:

$\vec{E}_{tangential} = 0$

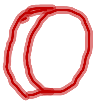
$\vec{E}_{normal} \neq 0$

Wire conductor:

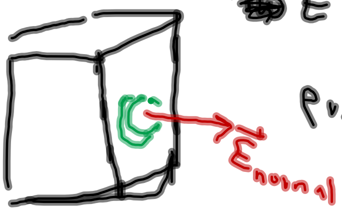
$\vec{E}_{tangential} = 0$

$\vec{E}_{normal} \neq 0$

Solid conductor:



"Gaussian pill-box"



$\vec{E} = 0$ inside

$\rho_{vol} = \nabla \cdot (\vec{0}) = 0$

outflux $= E_{normal}^{(outside)} \cdot area + 0^{inside} \cdot area$

+ practically zero through thin side.

$= \frac{1}{\epsilon_0}$ charge enclosed

$= \frac{1}{\epsilon_0}$ surface charge density × area

$$\Rightarrow \boxed{E_{normal} = \rho_s / \epsilon_0}$$

Interface conditions.

Back to finite differences.

$$FD: \quad f'(x_0) \simeq \frac{f(x_0+\Delta x)-f(x_0)}{\Delta x} + O(\Delta x)$$

$$BD \quad \ldots\ldots \quad \frac{\ldots}{-\Delta x} \quad + O(\Delta x)$$

$$CD = \frac{f(x_0+\Delta x)-f(x_0-\Delta x)}{2\Delta x} + O(\Delta x^2)$$

Good for $\vec{E} = -\nabla V$ $\quad \rho = \nabla \cdot (\epsilon_0 \vec{E})$

BUT to find the potential V,

we have to attack $\nabla^2 V = -\rho/\epsilon_0$

$$\nabla^2 V = \frac{\partial^2}{\partial x^2} V + \frac{\partial^2}{\partial y^2} V$$

We need Finite Difference for 2nd derivative.

Use these to get $f''(x_0)$

FD: $f'(x_0) \simeq \dfrac{f(x_0+\Delta x) - f(x_0)}{\Delta x} + \mathcal{O}(\Delta x)$

BD . - - - - . . - - $-\Delta x$ - $+ \mathcal{O}(\Delta x)$

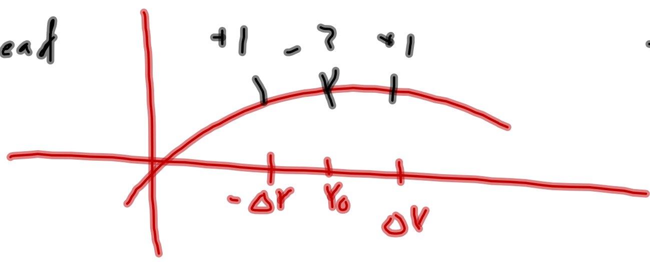
CD = $\dfrac{f(x_0+\Delta x) - f(x_0-\Delta x)}{2\Delta x} + \mathcal{O}(\Delta x^2)$

$$f''(x_0) = \frac{d}{dx} f'(x_0) = \frac{f'(x_0+\Delta x) - f'(x_0-\Delta x)}{2\Delta x} + \mathcal{O}(\Delta x^2)$$

$$= \frac{\dfrac{f(x_0+2\Delta x) - f(x_0)}{2\Delta x} - \dfrac{f(x_0) - f(x_0-2\Delta x)}{2\Delta x}}{2\Delta x}$$

$$= \frac{f(x_0+2\Delta x) - 2f(x_0) + f(x_0-2\Delta x)}{4\Delta x^2} + \mathcal{O}(\Delta x^2)$$

Instead

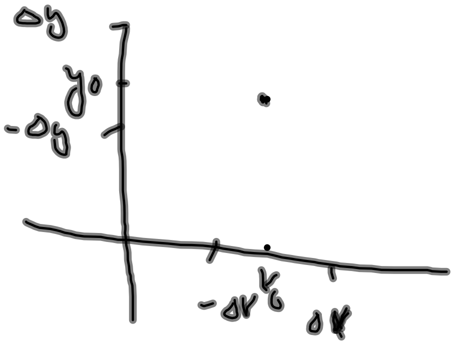


$$f''(x_0) = \frac{f(x_0+\Delta x) - 2f(x_0) + f(x_0-\Delta x)}{\Delta x^2}$$

interpret "$2\Delta x$" as "Δx"

CD $\quad f''(x_0) = \dfrac{f(x_0+\Delta x) - 2f(x_0) + f(x_0-\Delta x)}{\Delta x^2} + O(\Delta x^2)$

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} \simeq \frac{V(x_0+\Delta x, y_0) - 2V(x_0,y_0) + V(x_0-\Delta x, y_0)}{\Delta x^2}$$



$$+ \frac{V(x_0, y_0+\Delta y) - 2V(x_0,y_0) + V(x_0, y_0-\Delta y)}{\Delta y^2}$$

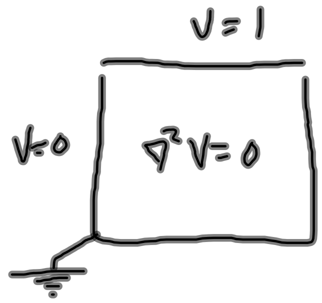
Take $\Delta x = \Delta y$

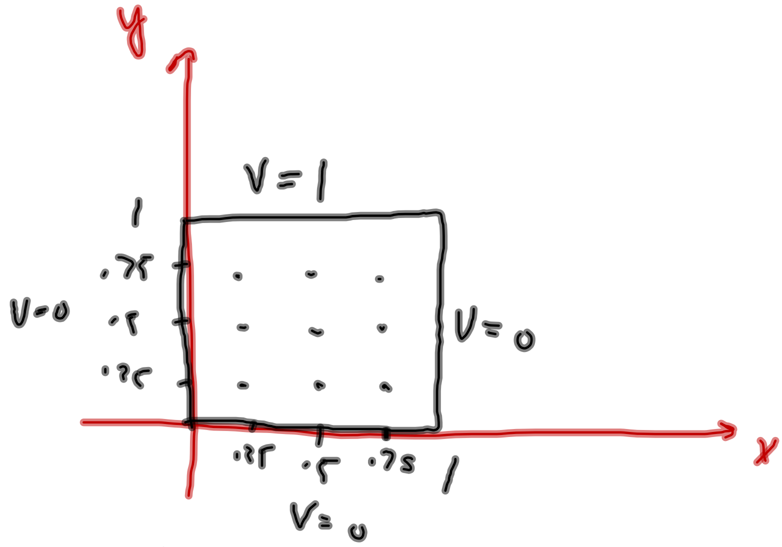
$$\nabla^2 V = \frac{V_\uparrow + V_\downarrow + V_\leftarrow + V_\rightarrow - 4V_0}{\Delta x^2}$$

"Laplace molecule)

$$\begin{array}{ccc} & +1 & \\ +1 & -4 & +1 \\ & +1 & \end{array}$$

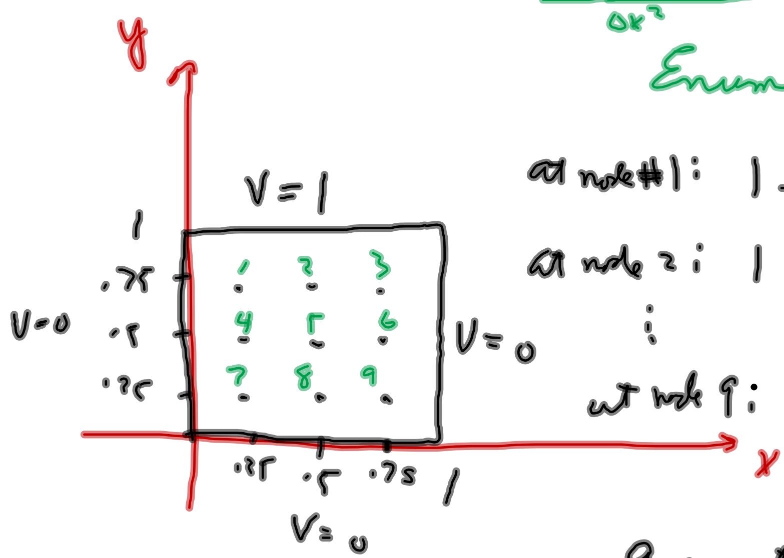
Formulate





$\Delta x = \Delta y = 0.25$

$$\nabla^2 V = 0 \approx \frac{V_\uparrow + V_\downarrow + V_\leftarrow + V_\rightarrow - 4V_0}{\Delta x^2}$$



$\Delta x = \Delta y = 0.25$

Enumerate $V_1, V_2, V_3, \ldots, V_9$

at node #1: $1 + V_4 + 0 + V_2 - 4V_1 = 0$

at node 2: $1 + V_5 + V_1 + V_3 - 4V_2 = 0$

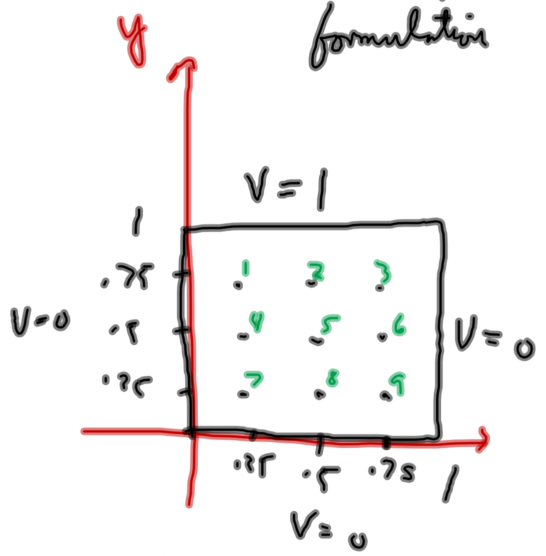
$\vdots$

at node 9: $- \_\_\_\_\_\_ = 0$

9 equations, 9 unknowns:

Isn't math marvelous?

matrix formulation



$\Delta x = \Delta y = 0.25$

$$1 + V_4 + 0 + V_2 - 4V_1 = 0$$

$$\begin{bmatrix} -4 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -4 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -4 & 0 & 0 & 1 & 0 & 0 & 0 \\ & & \vdots & & & & & & \\ & & \vdots & & & & & & \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ \vdots \\ V_9 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ -1 \\ \\ \\ \end{bmatrix}$$

Now ask matlab to solve $Ax = b$.

$\Rightarrow V \Rightarrow \vec{E} \Rightarrow \rho$

Problem: too big for matrix techniques.
Try iterative approach.

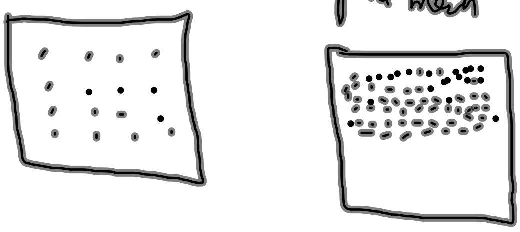
Laplace molecule: $V_{\uparrow} + V_{\downarrow} + V_{\leftarrow} + V_{\rightarrow} - 4V_0 = 0$

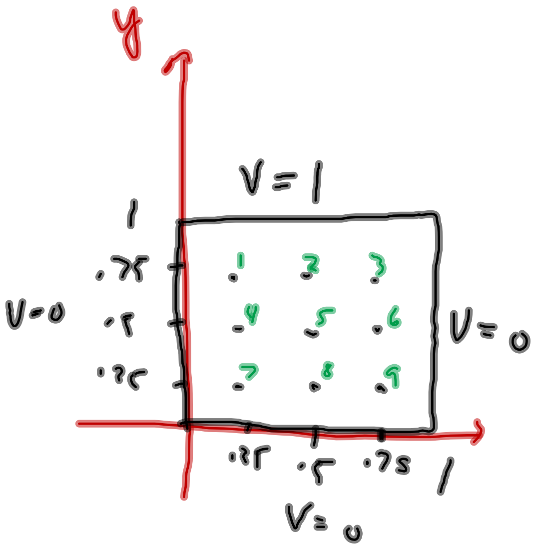
$$\Longrightarrow V_0 = \frac{V_{\uparrow} + V_{\downarrow} + V_{\leftarrow} + V_{\rightarrow}}{4}$$

center value = average of 4 neighbor

coarse mesh

fine mesh





Can we take advantage?

Reset V_{center} = average

Try to solve iteratively.

Start with all unknowns = 0.

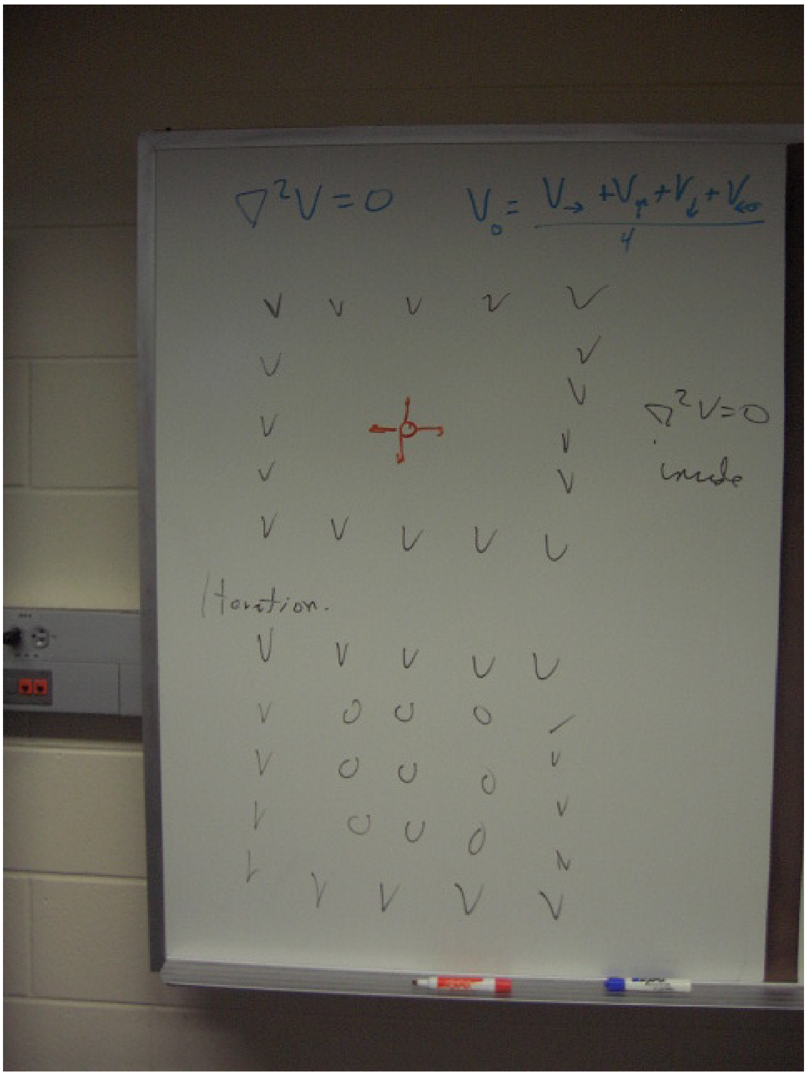
Re-set each value to the average of its neighbors.

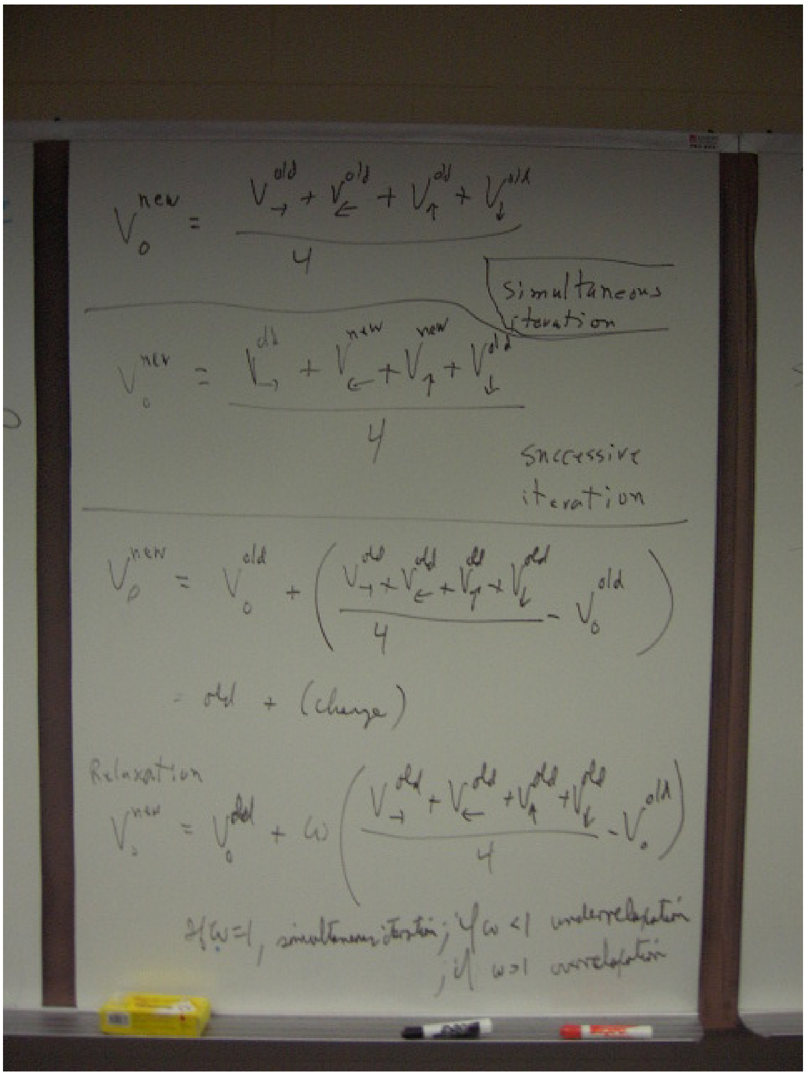
Repeat, ..., until no change.

This is called "relaxation."

If you use ~~new~~ updated values for the neighbors ↑ & ←, converge faster

called "Successive Relaxation".


∇²V = 0
V₀ = (V→ + V↑ + V← + V↓) / 4
∇²V = 0
inside
Iteration.



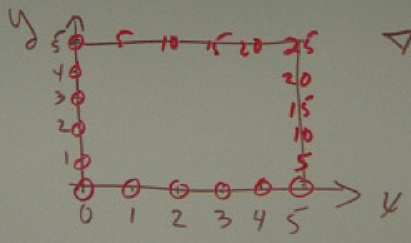
$V_0^{new} = \dfrac{V_\rightarrow^{old} + V_\leftarrow^{old} + V_\uparrow^{old} + V_\downarrow^{old}}{4}$
simultaneous iteration
$V_0^{new} = \dfrac{V_\rightarrow^{old} + V_\leftarrow^{new} + V_\uparrow^{new} + V_\downarrow^{old}}{4}$
successive iteration
$V_0^{new} = V_0^{old} + \left(\dfrac{V_\rightarrow^{old} + V_\leftarrow^{old} + V_\uparrow^{old} + V_\downarrow^{old}}{4} - V_0^{old} \right)$
= old + (change)
Relaxation
$V_0^{new} = V_0^{old} + \omega \left(\dfrac{V_\rightarrow^{old} + V_\leftarrow^{old} + V_\uparrow^{old} + V_\downarrow^{old}}{4} - V_0^{old} \right)$
If $\omega=1$, simultaneous iteration; if $\omega<1$ underrelaxation
; if $\omega>1$ overrelaxation

Simultaneous Relaxation:

$$V_0^{new} = \omega \frac{V_\uparrow^{old} + V_\downarrow^{old} + V_\rightarrow^{old} + V_\leftarrow^{old}}{4} + (1-\omega) V_0^{old}$$

Successive Overrelaxation "SOR"

$$V_0^{new} = \omega \frac{V_\uparrow^{new} + V_\downarrow^{old} + V_\rightarrow^{old} + V_\leftarrow^{new}}{4} + (1-\omega) V_0^{old}$$



$\nabla^2 V = 0$

answer $= V(x,y) = xy$

Check: $\nabla^2 V = 0$ ✓

Boundary Values

I argued that

$$\nabla^2 V = \frac{V_\uparrow + V_\rightarrow + V_\downarrow + V_\leftarrow - 4V_a}{4}$$

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + O(\Delta x^2)$$

In one-dimension:

$$\frac{\partial^2 V}{\partial x^2} = \frac{V(x+\Delta x) + V(x-\Delta x) - 2V(x)}{\Delta x^2} + O(\Delta x^2)$$

Analysis: ~~Sat:~~ Goal: to construct an approximation of $\frac{\partial^2 V}{\partial x^2}$ using $V(x)$, $V(x+\Delta x)$, $V(x-\Delta x)$, as accurate as possible.

Taylor series:

$$V(x) = V(x)$$

$$V(x+\Delta x) = V(x) + V'(x)\,\Delta x + \frac{V''(x)\,\Delta x^2}{2!} + \frac{V'''}{3}$$

$$V(x-\Delta x) = V(x) + V'(x)(-\Delta x) + \frac{V''(x)}{2!}(-\Delta x)^2 + \frac{V'''}{3!}$$

To approximate $V''(x)$ using

$$A\,V(x) + B\,V(x+\Delta x) + C\,V(x-\Delta x)$$

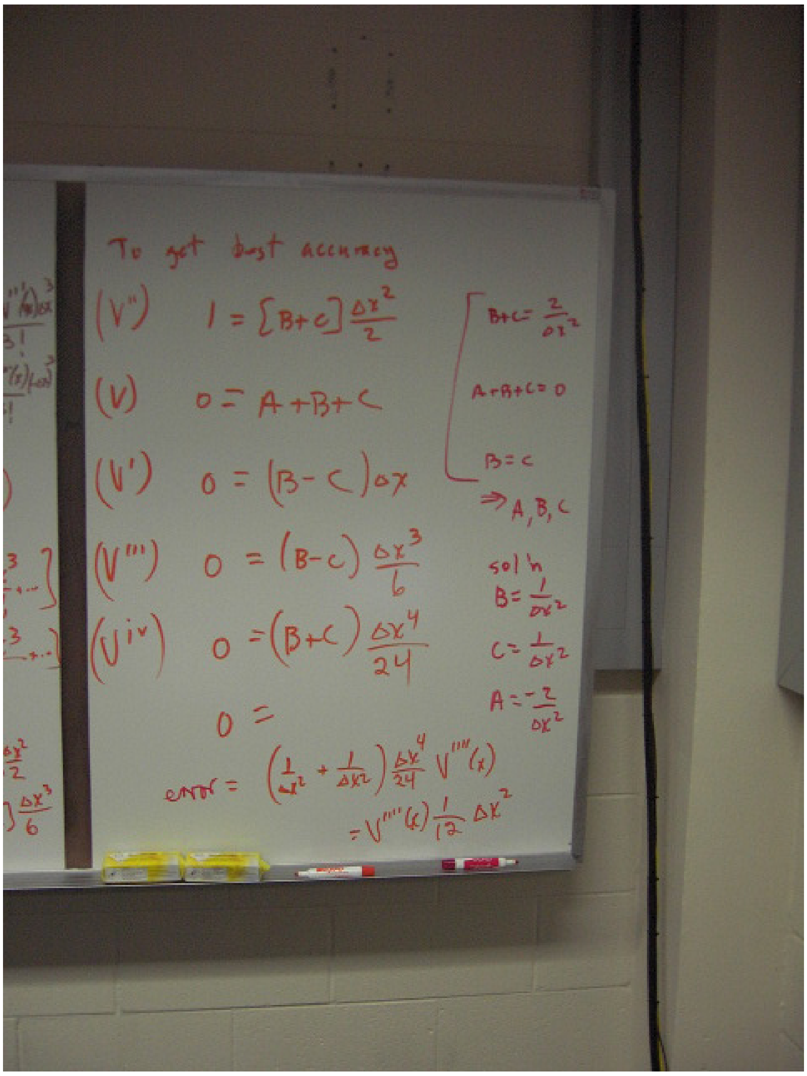
$$= A\,V(x) + B\left[V(x) + V'(x)\,\Delta x + V''\frac{\Delta x^2}{2!} + V'''\frac{\Delta x^3}{3!} + \dots\right.$$

$$+ C\left\{V(x) - V'\Delta x + V''\frac{\Delta x^2}{2!} - V'''\frac{\Delta x^3}{3!} + \dots\right.$$

$$= V''(x) + \text{error}$$

$$= V(x)\{A+B+C\} + V'[B-C]\,\Delta x + V''\{B+C\}\frac{\Delta x^2}{2}$$

$$+ V^{(v)}\dots \quad + V''''[B+C]\frac{\Delta x^4}{4!} + V'''[B-C]\frac{\Delta x^3}{6}$$

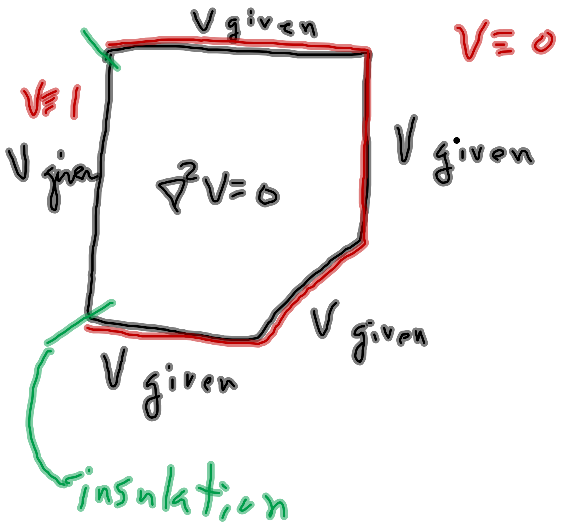

To get best accuracy
(V'') $1 = [B+C]\frac{\Delta x^2}{2}$
(V) $0 = A+B+C$
(V') $0 = (B-C)\Delta x$
(V''') $0 = (B-C)\frac{\Delta x^3}{6}$
(V^{iv}) $0 = (B+C)\frac{\Delta x^4}{24}$
$0 =$
error $= \left(\frac{1}{\Delta x^2} + \frac{1}{\Delta x^2}\right)\frac{\Delta x^4}{24} V''''(x)$
$= V''''(x)\frac{1}{12}\Delta x^2$
$B+C = \frac{2}{\Delta x^2}$
$A+B+C = 0$
$B = C$
$\Rightarrow A, B, C$
sol'n
$B = \frac{1}{\Delta x^2}$
$C = \frac{1}{\Delta x^2}$
$A = \frac{-2}{\Delta x^2}$

$$f'(x) = \frac{f(x+\Delta x) - f(x)}{\Delta x} + \underbrace{O(\Delta x)}_{f''(x)\frac{\Delta x}{2} + \cdots}$$

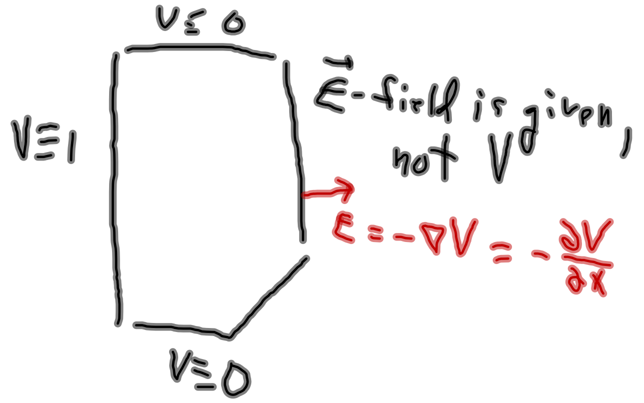
$$= \frac{f(x+\Delta x) - f(x-\Delta x)}{2\Delta x} + \underbrace{O(\Delta x^2)}_{\frac{f'''(x)\Delta x^2}{6!} + \cdots}$$

If $f(x) = \underline{x^5}$

$$f(x) = \underbrace{f(0)}_{0} + \underbrace{f'(0)}_{0}x + \underbrace{f''(0)}_{0}\frac{x^2}{2!} + \underbrace{\frac{f'''(0)x^3}{3!}}_{0} + \underbrace{\frac{f''''(0)x^4}{4!}}_{0} + \underbrace{\frac{f^{(5)}(0)x^5}{5!}}_{\neq 0}$$



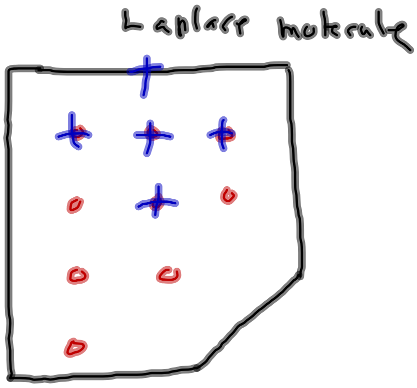
V given on boundary, called DIRICHLET boundary condition.

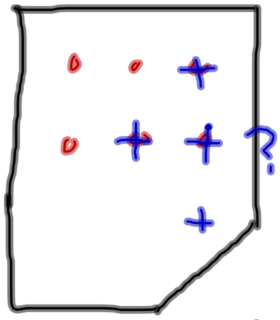


$\frac{\partial V}{\partial n_{ormal}}$ is given, (not V), called

NEUMANN

boundary condition.





Can't implement Laplace molecule at a point next to the boundary if it's not a Dirichlet boundary.

Instead, at this point write the FD equation expressing the Neuman condition:

$$\frac{\partial V}{\partial x} = (\text{Known}) \text{ at boundary.}$$

If x is on boundary, $\dfrac{V(x) - V(x-\Delta x)}{\Delta x} \simeq (\text{Known})$

Problem: accuracy is $O(\Delta x)$; everything is $O(\Delta x^2)$.

Cure: use a more accurate approximation for $\frac{\partial V}{\partial x}$.

Centered difference? NO

$$A\, V(x) + B\, V(x-\Delta x) + C\,(x-2\Delta x) \simeq \frac{\partial V}{\partial x}(x) + O(\Delta x^2)$$

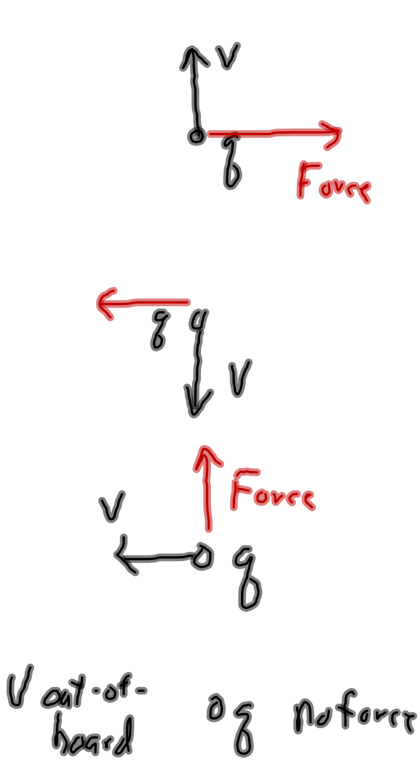

Write Laplace molecule eq.
Write Lap. mole. equation
Write FD equation enforcing the Neumann condition.

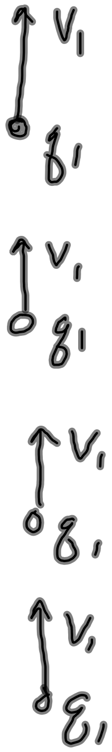
Magnetics is tough!

Force $\leftarrow q$ $\quad$ q_1 $\quad \leftarrow$ Coulomb

$\leftarrow$ $\quad$ q_1

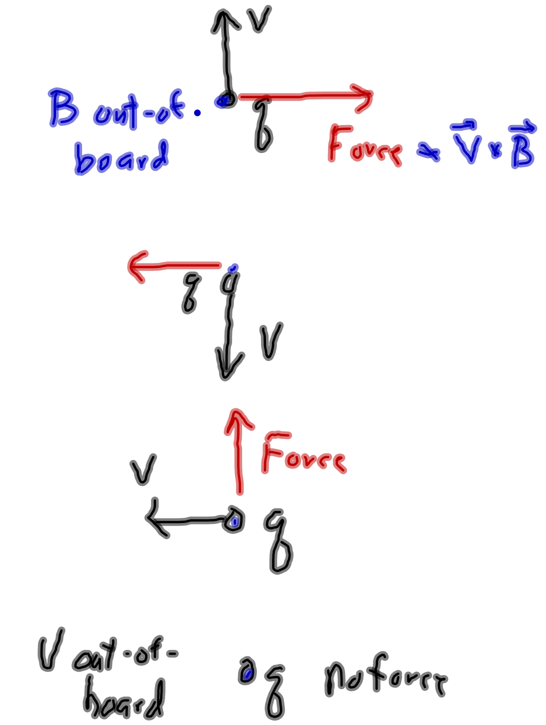
$E = F/q$

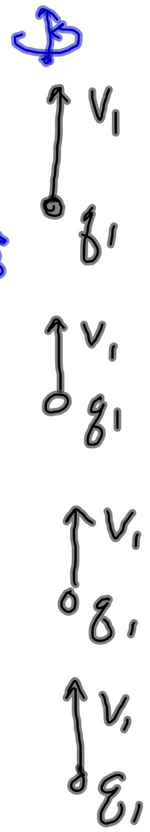




Ampere
Oersted
Biot
Savart
Lorentz

Magnetic ~~Field~~ $\vec{B}$
Flux Density





Ampere

Oersted

Biot

Savart

Lorentz

Force law:

$$\vec{F} = q\vec{V} \times \vec{B}$$

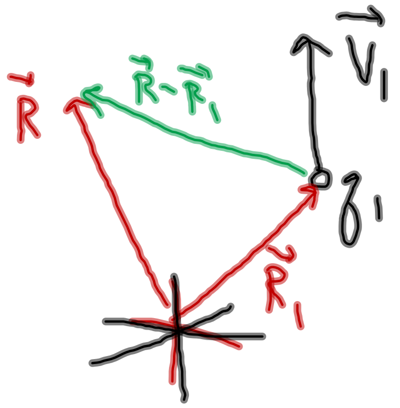
We have the formula for the <u>force</u> in terms of field $\vec{B}$.

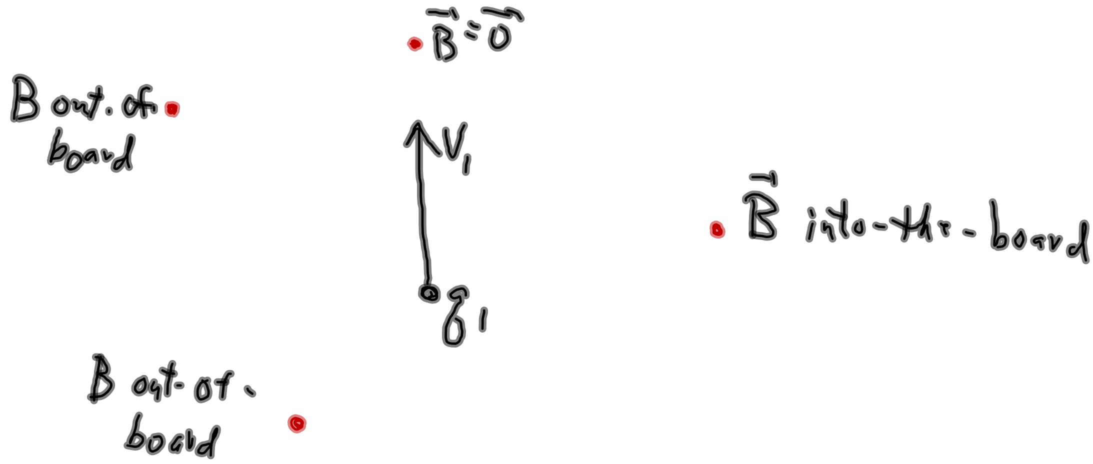
What's the formula for $\vec{B}$?

Biot-Savart

$$\vec{B} = \frac{q_1}{4\pi\mu_0} \frac{\vec{V}_1 \times (\vec{R}-\vec{R}_1)}{|\vec{R}-\vec{R}_1|^3}$$

I made an error. μ_0 goes in the numerator, not the denominator!

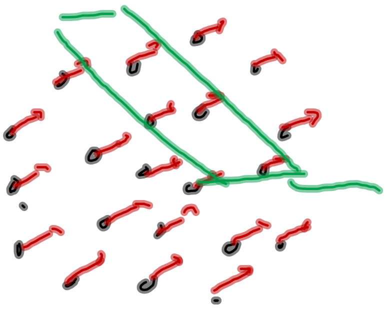




Book Keeping: current & individual charge motions.

Individual charge = q

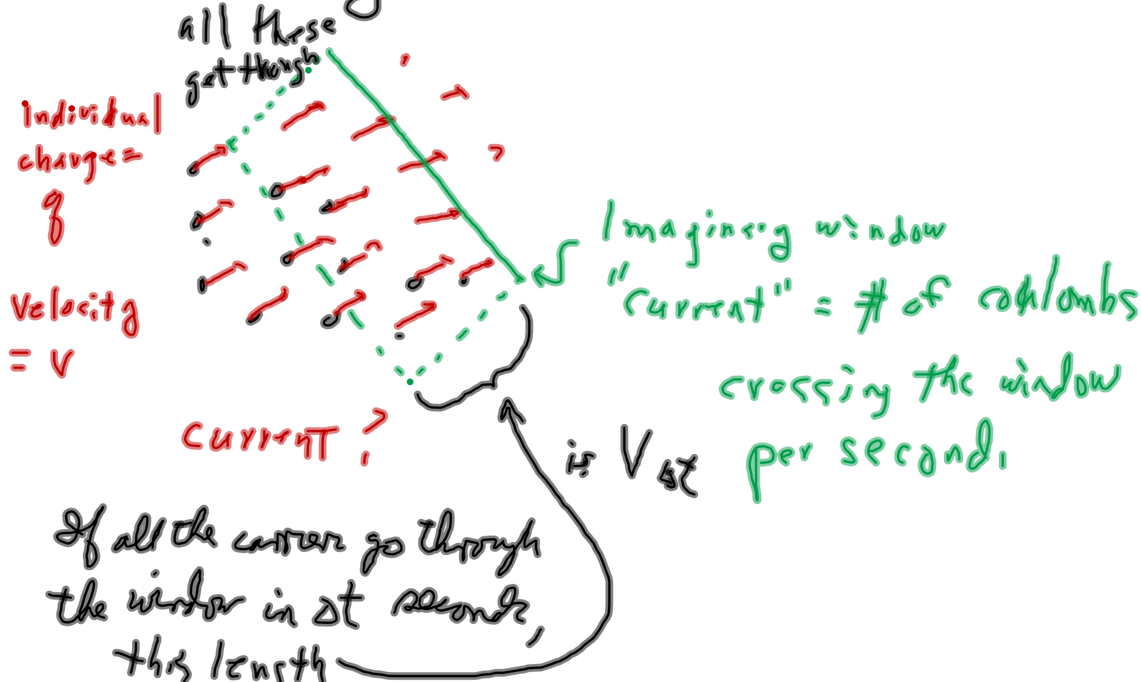
Velocity = v



Imagining window
"current" = # of coulombs crossing the window per second.

current ?

Book Keeping: Current & individual charge motions.



Every carrier in that volume passes through the window in time Δt.

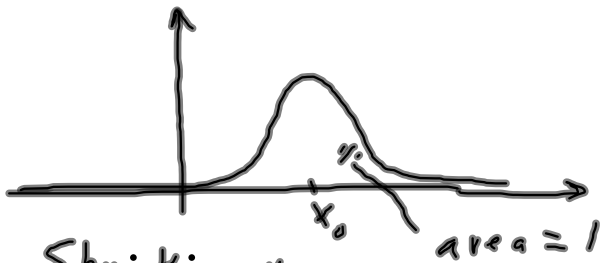
What is volume? $(A_{rea})(V\Delta t)$

" " charge in this volume? $\underbrace{\rho_v}_{\text{volume charge density}} A\, v\Delta t$

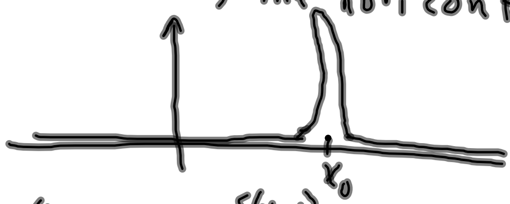
$$\text{Current} = \frac{\text{Charge}}{\Delta t} = \rho_v V A = \text{"}I\text{"}$$

We call $\rho_v V$ = "current per unit window area" = current density = "$\vec{J}$"

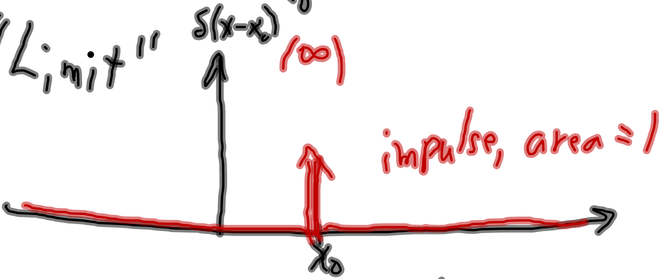
(Math) delta-function



Shrinking the horizontal span, keeping area = 1



"Limit"



$$\int_{-\infty}^{\infty} \delta(x-x_0) = 1 \qquad \int_a^b \delta(x-x_0) = \begin{cases} 1 & \text{if } a < x < b \\ 0 & \text{otherwise} \end{cases}$$

$$\int_{-\infty}^{\infty} \delta(x-x_0) f(x)\, dx = f(x_0) \int_{-\infty}^{\infty} \cdots = f(x_0)$$

Charge Distributions

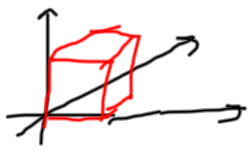
Point charge • q Coulombs

Volume charge density $\rho_v \, d(\text{volume})$

Surface " " $\rho_s \, d(\text{area})$ Coulombs

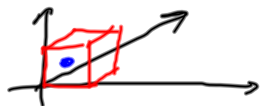
Line " " $\rho_L \, d(\text{length})$ Coulombs

The total charge enclosed in a volume is $\iiint \rho_v \underbrace{d\,vol}_{dx\,dy\,dz}$.



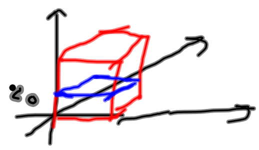
A point charge q_i Coulombs corresponds to a volume charge density $\rho_v = q_i\, \delta(x - x_i)\, \delta(y - y_i)\, \delta(z - z_i)$

because



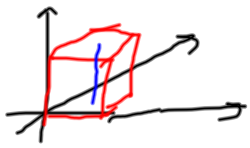
$$\iiint \underbrace{q_i\, \delta(x-x_i)\, \delta(y-y_i)\, \delta(z-z_i)}_{\rho_v}\, dx\, dy\, dz = q_i$$

A surface charge density ρ_s Coulomb/m^2 lying, say, in the plane $z = z_0$ corresponds to a volume charge density $\rho_v = \rho_s \, \delta(z - z_0)$



because $\iiint \underbrace{\rho_s \, \delta(z - z_0)}_{\rho_v} \, dx \, dy \, dz = \iint \rho_s \, dx \, dy$ = charge (Coulombs)

A line charge density ρ_L Coulombs/m lying, say, along the line $x = x_0$, $y = y_0$, parallel to the z axis, corresponds to a volume charge density $\rho_v = \rho_L \, \delta(x - x_0) \, \delta(y - y_0)$



because

$$\iiint \underbrace{\rho_L \, \delta(x - x_0) \, \delta(y - y_0)}_{\rho_v} \, dx \, dy \, dz = \int \rho_L \, dz$$

= charge (Coulombs)

Coulomb's Law:

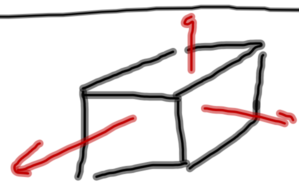
$$\vec{E}(\vec{R}) = \left\{ \begin{array}{c} \sum_i \frac{q_i}{4\pi\epsilon_0 |\vec{R}-\vec{R}_i|^2} \frac{\vec{R}-\vec{R}_i}{|\vec{R}-\vec{R}_i|} \\ + \\ \iiint \frac{\rho_V \, d(\text{volume}')}{4\pi\epsilon_0 |\vec{R}-\vec{R}'|^2} \frac{\vec{R}-\vec{R}'}{|\vec{R}-\vec{R}'|} \\ + \\ \iint \frac{\rho_S \, d(\text{area}')}{4\pi\epsilon_0 |\vec{R}-\vec{R}'|^2} \frac{\vec{R}-\vec{R}'}{|\vec{R}-\vec{R}'|} \\ + \\ \int \frac{\rho_L \, d(\text{length}')}{4\pi\epsilon_0 |\vec{R}-\vec{R}'|^2} \frac{\vec{R}-\vec{R}'}{|\vec{R}-\vec{R}'|} \end{array} \right\} \overset{\text{math}}{=} -\nabla V(\vec{R}) = -\nabla \left\{ \begin{array}{c} \sum_i \frac{q_i}{4\pi\epsilon_0 |\vec{R}-\vec{R}_i|} \\ + \\ \iiint \frac{\rho_V \, d(\text{vol}')}{4\pi\epsilon_0 |\vec{R}-\vec{R}'|} \\ + \\ \iint \frac{\rho_S \, d(\text{area}')}{4\pi\epsilon_0 |\vec{R}-\vec{R}'|} \\ + \\ \int \frac{\rho_L \, d(\text{length}')}{4\pi\epsilon_0 |\vec{R}-\vec{R}'|} \end{array} \right\}$$

math: $\vec{E} = -\nabla V$ so $\nabla \times \vec{E} = \vec{0}$

$$\nabla \times \vec{E} = -\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial V}{\partial x} & \frac{\partial V}{\partial y} & \frac{\partial V}{\partial z} \end{vmatrix}$$

$$= -\vec{i}\left(\frac{\partial}{\partial y}\frac{\partial V}{\partial z} - \frac{\partial}{\partial z}\frac{\partial V}{\partial y}\right) + \cdots$$

$$= \vec{0}$$

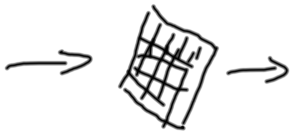


math:

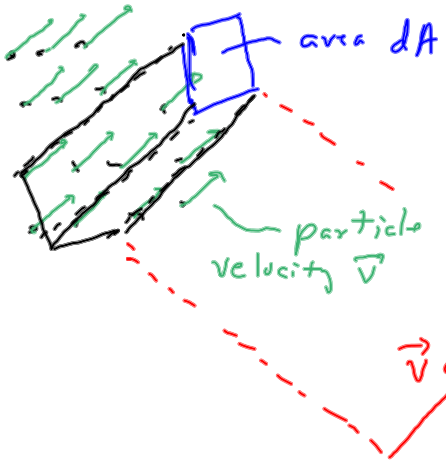
$$\iint \vec{E} \cdot \vec{n}_{\text{ormal}} \, d(\text{area}) = \frac{\text{charge enclosed}}{\epsilon_0}$$

$$\vec{\nabla} \cdot \vec{E} = \begin{array}{l}\text{outflux} \\ \text{per unit} \\ \text{volume}\end{array} = \frac{\text{charge per unit volume}}{\epsilon_0} = \frac{\rho_V}{\epsilon_0}$$

Current & current distribution.



The number of Coulombs per second passing through the area dA is the <u>current</u> I (C/sec) through A.



In time dt, all the charge in this volume dA by $|\vec{v}|\,dt$ passes through the window dA

$$\text{Volume} = \text{base area} \times \text{perpendicular height}$$

$$= \underbrace{dA}_{} \times \underbrace{|\vec{v}|\,dt \times \cos\theta}_{} = dA\, \vec{n}_{\text{ormal}} \cdot \vec{v}\, dt$$

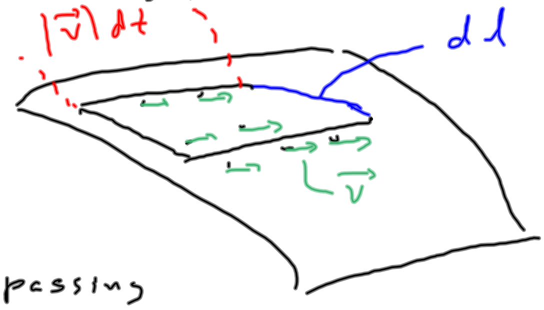
$$\text{Charge} = \rho_v \times \text{volume}$$

$$\text{Current} = \text{charge per second} = \frac{\rho_v \cdot \text{volume}}{dt} = \rho_v \vec{v} \cdot \vec{n}\, dA$$

So to get the current through a "large" window, integrate

$$I = \iint_{\text{area}} \underbrace{\rho_v \vec{v}}_{\text{"current charge density"} \equiv \vec{j}_v} \cdot \vec{n}\, dA$$

For surface charge,



the charge passing through line dl in time dt equals all the charge in the patch dl-by-$|\vec{v}\,dt|$.

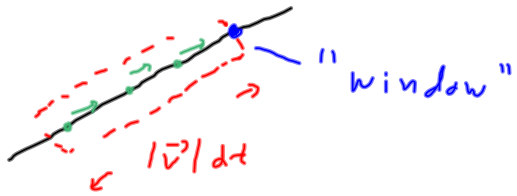
Charge $= \rho_s \, d\text{area} = \rho_s \, dl \, |\vec{v}| \, dt \cos\theta = \rho_s |\vec{v}| \, dl \cos\theta \, dt$

Current $= |\rho_s \vec{v} \times \vec{n}_{\text{ormal}}| \, dl$

Total current through a "long" line in the surface equals

$$\int_{\text{line}} |\underbrace{\rho_s \vec{v}}_{\text{surface current density } \vec{J}_s} \times \vec{n}| \, dl$$

For line current density,

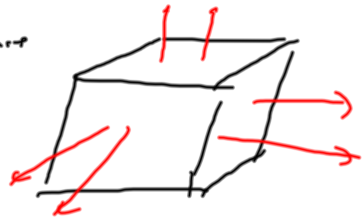


in time dt all the charge in wire of length $|\vec{v}|dt$ passes the window "dot".

Charge $= \rho_L |\vec{v}| dt$ current $= \rho_L |\vec{v}| = I$

Vector Analysis Theorems.

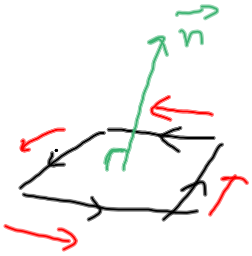
Divergence Theorem



Outflux of $\vec{F}$ per unit volume

$\to \vec{\nabla}\cdot\vec{F}$

Stokes's Theorem



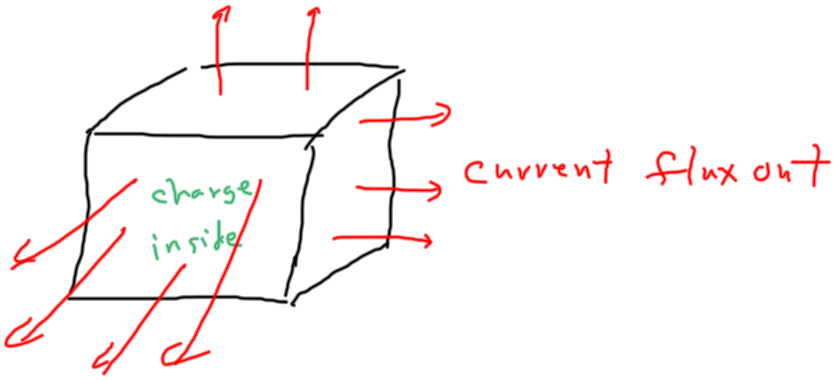
$\int \vec{F}\cdot d\vec{R}$ per unit area

$\to \vec{\nabla}\times\vec{F}\cdot\vec{n}$

Scalar potential $\vec{F} = \vec{\nabla}\phi \iff \nabla\times\vec{F} = \vec{0}$ e.g., $\vec{E} = \nabla(-V)$

Vector potential $\vec{F} = \nabla\times\vec{G} \iff \nabla\cdot\vec{F} = 0$

Conservation of charge



The current leaving the box = charge-per-second leaving the box

= rate of decrease of charge enclosed.

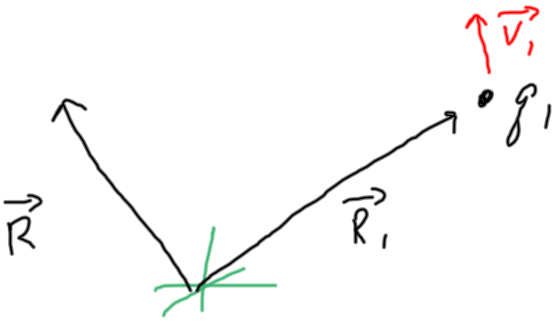
$$\iint \underbrace{\vec{J}}_{(\rho_v \vec{v})} \cdot \vec{n} \, d\text{area} = -\frac{d}{dt} \iiint \rho_v \, d\text{volume}$$

PER UNIT VOLUME THIS SAYS

$$\vec{\nabla} \cdot \vec{J} = -\frac{\partial \rho_v}{\partial t}.$$

"Continuity equation"

"Conservation of charge"



$$\vec{B}(\vec{R}) = \frac{q_1}{4\pi\mu_0} \frac{\vec{v}_1}{|\vec{R}-\vec{R}_1|^2} \times \frac{\vec{R}-\vec{R}_1}{|\vec{R}-\vec{R}_1|}$$

I made an error. μ_0 goes in the numerator, not the denominator!

$$\vec{B}(\vec{R}) = \left\{ \begin{array}{c} \displaystyle\sum_i \frac{q_i \vec{v}_i}{4\pi\mu_0} \times \frac{\vec{R}-\vec{R}_i}{|\vec{R}-\vec{R}_i|^3} \\ + \\ \displaystyle\iiint \frac{\rho_v \, d(\text{volume}') \, \vec{v}\,' \times (\vec{R}-\vec{R}\,')}{4\pi\mu_0 |\vec{R}-\vec{R}\,'|^3} \\ + \\ \displaystyle\iint \frac{\rho_s \, d(\text{area}') \, \vec{v}\,' \times (\vec{R}-\vec{R}\,')}{4\pi\mu_0 |\vec{R}-\vec{R}\,'|^3} \\ + \\ \displaystyle\int \frac{\rho_L \, d(\text{length}) \, \vec{v}\,' \times (\vec{R}-\vec{R}\,')}{4\pi\mu_0 |\vec{R}-\vec{R}\,'|^3} \end{array} \right\} = \left\{ \begin{array}{c} (\text{same}) \\ + \\ \displaystyle\iiint \frac{\vec{J}_v\,' \, d(\text{vol}') \times (\vec{R}-\vec{R}\,')}{4\pi\mu_0 |\vec{R}-\vec{R}\,'|^3} \\ + \\ \displaystyle\iint \frac{\vec{J}_s\,' \, d(\text{area})' \times (\vec{R}-\vec{R}\,')}{4\pi\mu_0 |\vec{R}-\vec{R}\,'|^3} \\ + \\ \displaystyle I \int \frac{d(\overrightarrow{\text{length}'}) \times (\vec{R}-\vec{R}\,')}{4\pi\mu_0 |\vec{R}-\vec{R}\,'|^3} \end{array} \right\}$$

$$\vec{B}(\vec{R}) = \left\{ \begin{array}{c} \sum_i \dfrac{q_i \vec{V}_i \times}{4\pi\mu_0} \dfrac{\vec{R}-\vec{R}_i}{|\vec{R}-\vec{R}_i|^3} \\ + \\ \iiint \dfrac{\rho_v \, d(\text{volume}') \, \vec{V}' \times (\vec{R}-\vec{R}')}{4\pi\mu_0 |\vec{R}-\vec{R}'|^3} \\ + \\ \iint \dfrac{\rho_s \, d(\text{area}') \, \vec{V}' \times (\vec{R}-\vec{R}')}{4\pi\mu_0 |\vec{R}-\vec{R}'|^3} \\ + \\ \int \dfrac{\rho_l \, d(\text{length}) \, \vec{V}' \times (\vec{R}-\vec{R}')}{4\pi\mu_0 |\vec{R}-\vec{R}'|^3} \end{array} \right\} = \left\{ \begin{array}{c} (\text{same}) \\ + \\ \iiint \dfrac{\vec{j}_v' \, d(\text{vol}')}{4\pi\mu_0 |\vec{R}-\vec{R}'|^3} \\ + \\ \iint \dfrac{\vec{j}_s' \, d(\text{area})'}{4\pi\mu_0 |\vec{R}-\vec{R}'|^3} \\ + \\ I \int \dfrac{d\,\overrightarrow{(\text{length}')}}{4\pi\mu_0 |\vec{R}-\vec{R}'|} \end{array} \right\}$$

math

$$= \nabla \times \vec{A}$$

where the "vector potential" is

I made an error. μ_0 goes in the numerator, not the denominator!

$$\vec{A} = \iiint \frac{\vec{j}_v \, d(\text{volume}')}{4\pi\mu_0 |\vec{R}-\vec{R}'|} + \cdots + I \int \frac{d\,\overrightarrow{\text{length}'}}{4\pi\mu_0 |\vec{R}-\vec{R}'|}$$

---

math: Since $\vec{B} = \vec{\nabla} \times \vec{A}$, $\vec{\nabla} \cdot \vec{B} = 0$

$$\nabla \cdot \vec{B} = \frac{\partial}{\partial x}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) - \frac{\partial}{\partial y}\left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right) + \frac{\partial}{\partial z}\left(\frac{\partial A_x}{\partial y} - \frac{\partial A_y}{\partial x}\right) = 0$$

Also math $\Rightarrow$

$$\nabla \times \vec{B} = \frac{\vec{j}_v}{\mu_0}$$

Remember the forces on a charge q with velocity $\vec{v}$ at position $\vec{R}$;

$$F = q\vec{E}(\vec{R}) + q\vec{v} \times \vec{B}(\vec{R})$$

"LORENTZ FORCE"

The force on a collection of charges $\rho_v(\vec{R})$ with velocities $\vec{v}(\vec{R})$ is the sum (integral)

$$\vec{F} = \iiint \left(\rho_v \vec{E} + \vec{j}_v \times \vec{B}\right) d\,\text{volume}$$

SUMMARY OF EQUATIONS

Lorentz Force

$$\vec{F} = q\vec{E} + q\vec{v}\times\vec{B}$$

Coulomb   Ampere

Conservation of Charge

$$\vec{\nabla}\cdot\vec{j} = -\frac{\partial\rho}{\partial t}$$

Electrostatics:

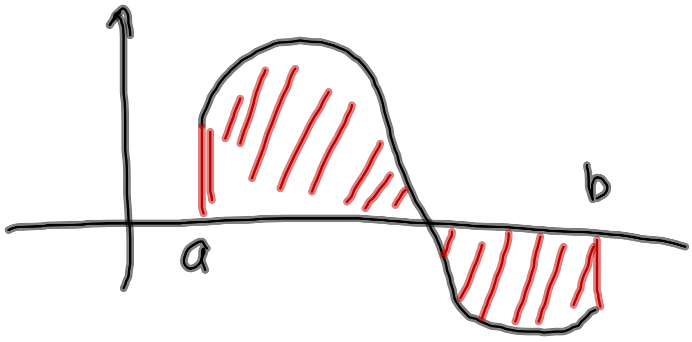
$$\nabla\cdot\vec{E} = \frac{\rho}{\epsilon_0} \qquad \nabla\times\vec{E} = \vec{0}$$

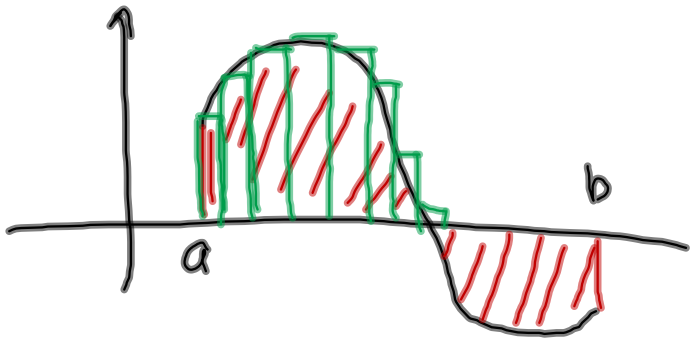
Magnetostatics:

$$\nabla\cdot\vec{B} = 0 \qquad \vec{\nabla}\times\vec{B} = \frac{\vec{J}_v}{\cancel{\mu_0}}$$

I made an error. μ_0 goes in the numerator, not the denominator!

How does a computer do integrals?

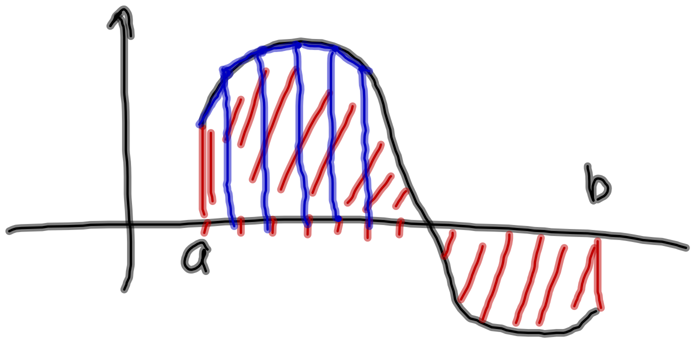




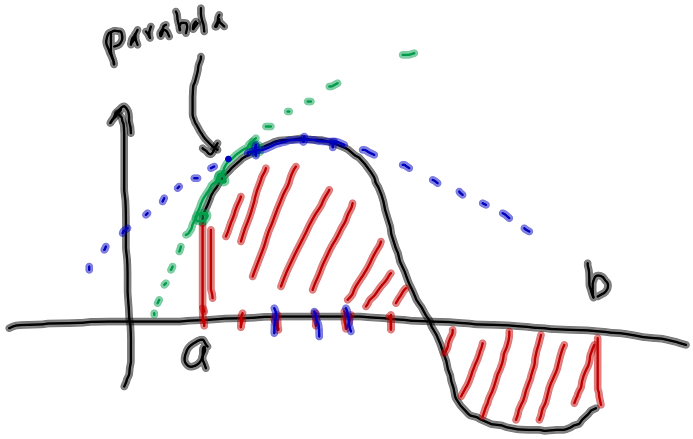
Rectangle construction.

(Riemann sum for the integral.)

more accurate



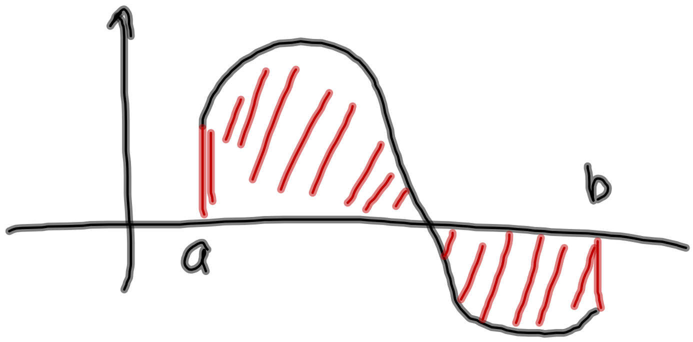
Trapezoid Rule



Simpson's Rule

① fit with parabolas

② $\int (ax^2+bx+c)\,dx$ easily.



Sampson's Rule

fit a cubic

$\int ax^3 + \cdots$

$$\nabla\cdot\vec{j}_v = -\frac{\partial\rho_v}{\partial t}$$ Conservation of Charge

outflux of current (per unit vol)

decrease of charge enclosed (p.u. volume)

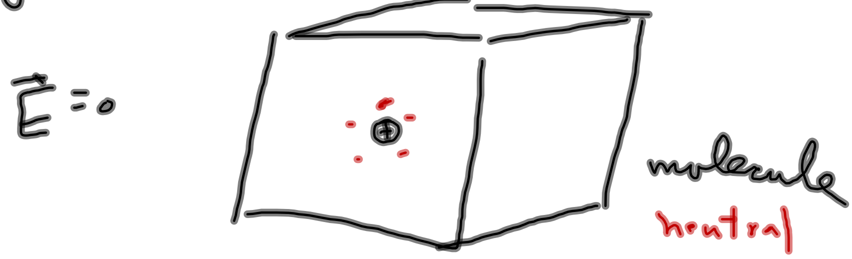
$$\nabla\cdot\vec{E} = \frac{\rho_v}{\epsilon_0}$$ (outflux of $\epsilon_0\vec{E}$ = charge enclosed)

$$\vec{\nabla}\times\vec{E} = 0$$ (line integrals of $\vec{E}$ around closed paths = 0)

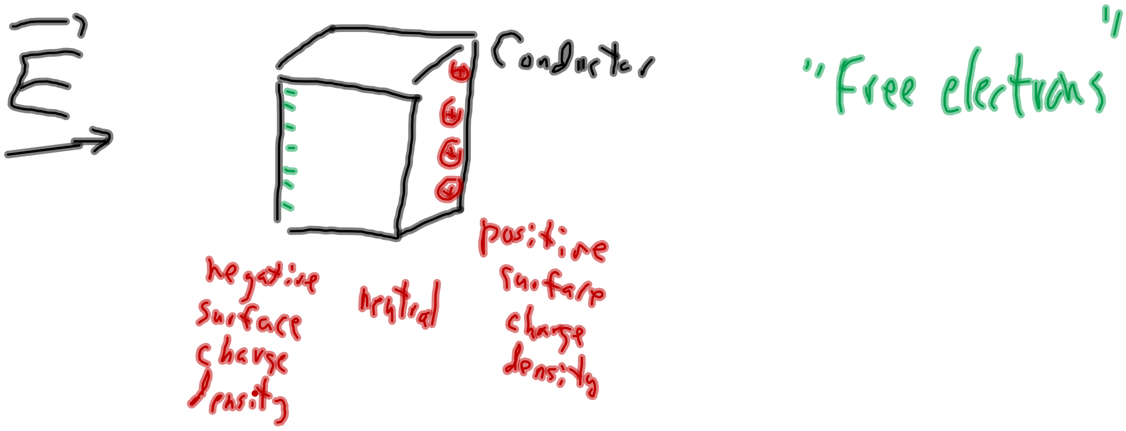
$$\nabla\cdot\vec{B} = 0$$ (by analogy with $\nabla\cdot\vec{E} = 0$, there are no magnetic point charges)

$$\vec{\nabla}\times\vec{B} = \cancel{\frac{\vec{j}_v}{\mu_0}} = \mu_0\vec{j}_v$$

If you're inside (dielectric, plasma, something)



The shift in bound charges (electrons & nucleus) contribute to the total electric field



How do we account for these 2 effects?

① For the conductor, use voltage, treat conductor as equipotential, solve P.D.E.

② For dielectric, distinguish between the actual electric field produced by all charge, and the "mathematical" " produced only by free charge.

The dipole in a dielectric is created by the external $\vec{E}$-field. "Polarizability."

If ~~it~~ we assume proportionality, then

$$\vec{D} = \epsilon \vec{E}$$ is the field due to free charges only.

$$\iint_{\text{surface}} \vec{E} \cdot \vec{n} \, d_{\text{area}} = \frac{\rho_v^{\text{total}}}{\epsilon_0} \qquad \underbrace{\iint_{\text{surface}} \vec{D} \cdot \vec{n} \, d_{\text{area}} = \frac{\rho_v^{\text{free}}}{\cancel{\epsilon_0}}}_{\vec{\nabla} \cdot \vec{D} = \rho_v^{\text{free}}}$$

If no bound charges, $\vec{D} = \epsilon_0 \vec{E}$

So $\vec{D} = \epsilon \vec{E}$ ~~repla~~ and the 2 electrostatic equations are

$$\vec{\nabla} \cdot \vec{D} = \rho_v^{free}$$

$$\vec{\nabla} \times \vec{E} = 0$$

Need to model magnetic materials the same way.

Assume the induced magnetization is proportional to the applied $\vec{B}$-field; create a mathematical field which is produced by free currents, not by induced magnetization.

$$\vec{H} = \mu \vec{B} \qquad \vec{D} = \epsilon \vec{E}$$

No! $\vec{H} = \frac{1}{\mu} \vec{B}$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}_v^{total} \Rightarrow \vec{\nabla} \times \vec{H} = \vec{J}_v^{free}, \quad \nabla \cdot \vec{B} = 0 \text{ (still)}$$

Cons. of charge: $\nabla \cdot \vec{j}_v^{total} = -\dfrac{\partial \rho_v^{total}}{\partial t}$

$$\nabla \cdot \vec{D} = \rho_v^{free} \qquad \nabla \times \vec{E} = \vec{0}$$

"Electric Displacement Vector", or "Flux Density" "Electric Field"

$$\vec{\nabla} \cdot \vec{B} = 0 \qquad \vec{\nabla} \times \vec{H} = \vec{j}_v^{free}$$

"Magnetic Flux Density" "Magnetic Field"

Ironically, most EM works use $\vec{E}$ & $\vec{H}$.

Time Dependence

Cons. of charge: $\nabla \vec{j}_v^{\,total} = -\frac{\partial \rho_v^{total}}{\partial t}$

$\nabla \cdot \vec{D} = \rho_v^{free}$ $\qquad$ $\nabla \times \vec{E} = \cancel{\vec{0}} - \frac{\partial \vec{B}}{\partial t}$ $\quad$!!!

"Electric Displacement Vector", or "Flux Density" $\qquad$ "Electric Field"

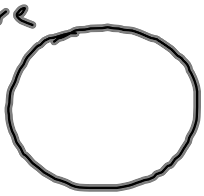
$\vec{\nabla} \cdot \vec{B} = 0$ $\qquad$ $\vec{\nabla} \times \vec{H} = \vec{j}_v^{\,free} + \frac{\partial \vec{D}}{\partial t}$

"Magnetic Flux Density" $\qquad$ "Magnetic Field"

Ironically, most EM works use $\vec{E}$ & $\vec{H}$.

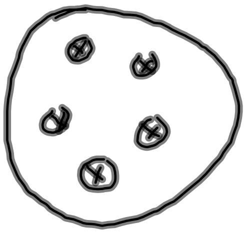
$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ Faraday Effect

wire with resistance



no current

Bring in a magnetic field



create a flux

You get a current as long as the flux changes.

Compare current & resistance, conclude $\oint \vec{E} \cdot d\vec{R} = -\frac{d\,\text{flux}}{dt}$

Equivalent to a voltage source $V = \oint \vec{E} \cdot d\vec{R}$.

PROBLEM:

vector analysis: $\vec{E}$ comes from a potential $\iff \nabla \times \vec{E} = \vec{0}$

So when $\frac{\partial \vec{B}}{\partial t} \neq 0$, lose the math. convenience of a potential

What to do? There are tricks, we will see.

$$V = \frac{d\,\text{flux}}{dt} = L\frac{dI}{dt}$$

This is one way of restoring the convenience of a potential.

Works only for Lumped-Parameter networks (low frequency)

$$\nabla \times \vec{H} = j_v^{free} + \frac{\partial \vec{D}}{\partial t}$$ Maxwell Displacement Current

~~Historically~~

If $\nabla \times \vec{H} = j_v^{free}$

take div $\nabla \cdot \nabla \times \vec{H} = \vec{\nabla} \cdot j_v^{free}$

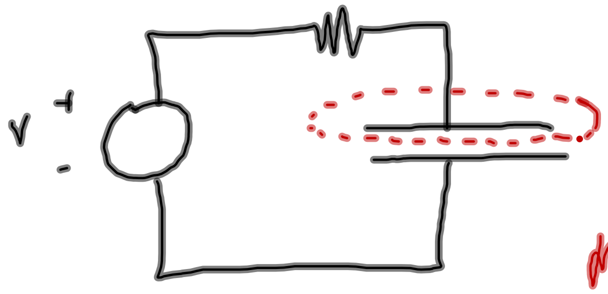
from vector analysis $\underbrace{\nabla \cdot \nabla \times \vec{H}}_{=0}$

$$\vec{\nabla} \cdot j_v^{free} = -\frac{\partial}{\partial t}\rho_v^{free} = -\frac{\partial}{\partial t}\vec{\nabla}\cdot\vec{D}$$

So repair the eq. by $\vec{\nabla} \times \vec{H} = j_v^{free} + \frac{\partial \vec{D}}{\partial t}$

Now div $0 = \nabla \cdot \vec{j} + \frac{\partial}{\partial t}\nabla \cdot D = \nabla \cdot \vec{j} + \frac{\partial \rho}{\partial t} = 0$

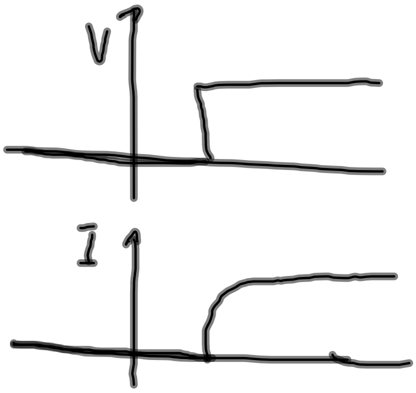
Historically.



Kirchhoff's current law is violated for ~~the~~ this "node"

Maxwell noticed that

$$I_{in} = \frac{\partial}{\partial t}\underbrace{\iint \vec{D}\cdot\hat{n}\,da_{open}}_{\text{between the plates.}}$$



$\frac{\partial \vec{D}}{\partial t}$ is 'equivalent' to a current density.

$$\nabla \times \vec{H} = \vec{J}_v^{\,free} + \frac{\partial \vec{D}}{\partial t}$$

$$\nabla\cdot\vec{j} = -\frac{\partial\rho}{\partial t} \qquad \nabla\cdot D = \rho \qquad \nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \qquad \nabla\cdot\vec{B} = 0$$

$$\nabla\times\vec{H} = \vec{j} + \frac{\partial\vec{D}}{\partial t}$$

This is implied by the others, so neglect it.

Just a little more math:

$$\nabla\times\frac{\vec{B}}{\mu} = \vec{j} + \frac{\partial\epsilon\vec{E}}{\partial t} \quad \cancel{\text{take}}$$

Take time derivative:

$$\frac{1}{\mu}\vec{\nabla}\times\frac{\partial\vec{B}}{\partial t} = (\text{assume } \vec{j}=0) + \epsilon\frac{\partial^2\vec{E}}{\partial t^2}$$

Insert Faraday's law

$$-\frac{1}{\mu}\underbrace{\vec{\nabla}\times\vec{\nabla}\times\vec{E}} = \epsilon\frac{\partial^2\vec{E}}{\partial t^2}$$

Vector identity

$$-\nabla^2\vec{E} + \vec{\nabla}(\underbrace{\nabla\cdot\vec{E}}_{\frac{\rho}{\epsilon}})$$

assume $\rho = 0$

$$\frac{1}{\mu}\vec{\nabla}^2\vec{E} = \epsilon\frac{\partial^2\vec{E}}{\partial t^2}$$

$$\frac{\partial\vec{E}}{\partial t^2} = \frac{1}{\epsilon\mu}\nabla^2\vec{E} \quad \text{wave equation}$$

For x-component

$$\frac{\partial^2 E_x}{\partial t^2} = \frac{1}{\epsilon\mu}\nabla^2 E_x$$

For x-component

$$\frac{\partial^2 E_x}{\partial t^2} = \frac{1}{\epsilon\mu} \nabla^2 E_x$$

Finite differences. E_x does not depend on y or z.

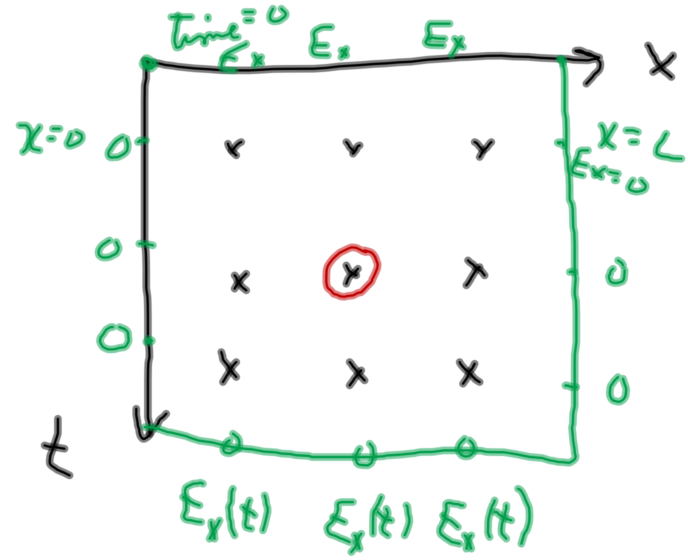
$$\frac{\partial^2 E_x}{\partial t^2} = \frac{1}{\epsilon\mu} \frac{\partial^2 E_x}{\partial x^2}$$

Compare with

$$\frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_x}{\partial \cancel{y}^2} = 0$$

$\frac{1}{\epsilon\mu}$ $\quad$ $-$ $\quad$ t

So consider EXCEL

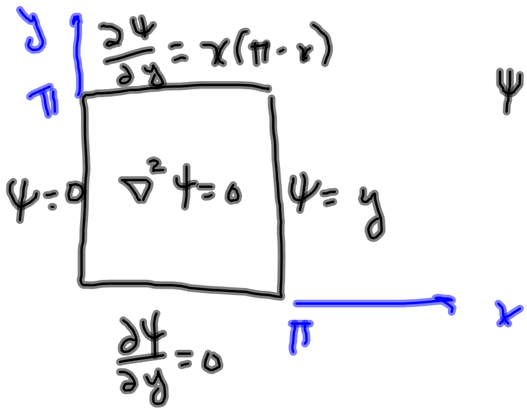


$$E_0 = \frac{E_\uparrow + E_\downarrow + E_\leftarrow + E_\rightarrow}{4}$$

Insert $\frac{1}{\epsilon\mu}$.

Minus signs

This data is not available. (It's the payoff.)



$$\psi = \frac{x}{2} - \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{\sinh(2n+1)x \;\; \cos(2n+1)y}{\sinh(2n+1)\pi \;\; (2n+1)^2}$$

$$+ \frac{8}{\pi} \sum_{n=0}^{\infty} \frac{\sin(2n+1)x \;\; \cosh(2n+1)y}{(2n+1)^4 \;\; \sinh(2n+1)\pi}$$

(p 269)

5 equations to solve

#1 : $\nabla^2 \psi = 0$: $\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$

$\nabla^2 x = 0$

$\nabla^2 \; \sinh(2n+1)x \cos(2n+1)y$

$= (2n+1)^2 \{\sinh\cos - \sinh\cos\}$

$= 0$

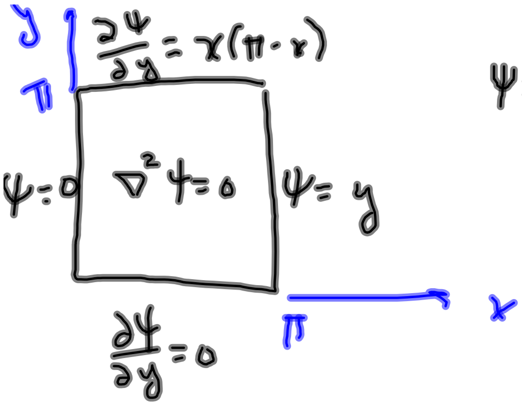
~~∇^2 (second last)~~

∇^2 (last term) $= 0$

similarly.

#1 is checked.

(It works term-by-term!)



$$\psi = \frac{x}{2} - \frac{4}{\pi}\sum_{n=0}^{\infty} \frac{\sinh(2n+1)x \;\; \cos(2n+1)y}{\sinh(2n+1)\pi \;\; (2n+1)^2}$$

$$+ \frac{8}{\pi}\sum_{n=0}^{\infty} \frac{\sin(2n+1)x \;\; \cosh(2n+1)y}{(2n+1)^4 \;\; \sinh(2n+1)\pi}$$

Eqn. #2: the left edge,

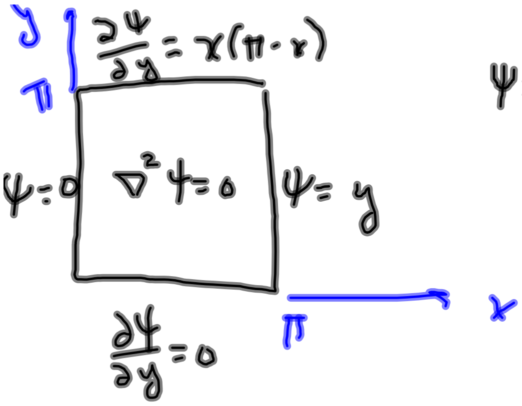
$$\psi(\underset{0}{\cancel{x}}, y) = 0 \qquad (x=0) = 0$$

Checks,

$$\sinh(2n+1)0 = 0$$

$$\sin(2n+1)0 = 0$$

It works term-by-term



$$\psi = \frac{x}{2} - \frac{4}{\pi}\sum_{n=0}^{\infty} \frac{\sinh(2n+1)x \quad \cos(2n+1)y}{\sinh(2n+1)\pi \quad (2n+1)^2}$$

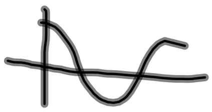
$$+ \frac{8}{\pi}\sum_{n=0}^{\infty} \frac{\sin(2n+1)x \quad \cosh(2n+1)y}{(2n+1)^4 \quad \sinh(2n+1)\pi}$$

#3 ~~Eqn~~ Eqn : bottom edge

$$\frac{\partial \psi(x, \overset{0}{y})}{\partial y} = 0$$

$$\frac{\partial (x)}{\partial y} = 0$$

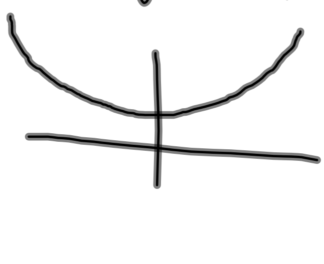
$\cos(2n+1)y$ is flat at 0

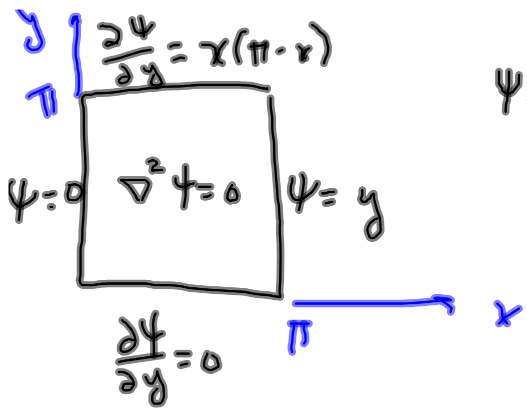


#3 checks.

It works term-by-term.

$\cosh(2n+1)y$ is flat at 0





$$\psi = \frac{x}{2} - \frac{4}{\pi}\sum_{n=0}^{\infty} \frac{\sinh(2n+1)x \quad \cos(2n+1)y}{\sinh(2n+1)\pi \quad (2n+1)^2}$$

$$+ \frac{8}{\pi}\sum_{n=0}^{\infty} \frac{\sin(2n+1)x \quad \cosh(2n+1)y}{(2n+1)^4 \quad \sinh(2n+1)\pi}$$

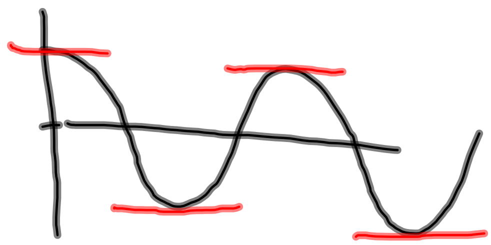
Eqn. #4 : top edge

$$\frac{\partial\psi}{\partial y}(x, \pi) = x(\pi - x)$$

$$\frac{\partial}{\partial y}(x) = 0$$

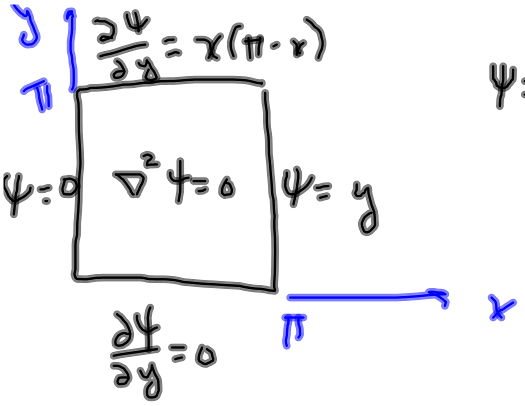
$$\left.\frac{\partial}{\partial y}\cos(2n+1)y\right|_{y=\pi} = \underbrace{-(2n+1)\sin(2n+1)\pi}_{=0}$$

cos is flat when it is ± 1!



So $\left.\frac{\partial}{\partial y}\cos(2n+1)y\right|_{y=\pi} = 0$

So the first line is flat at top, contributes zero to $\left.\frac{\partial\psi}{\partial y}\right|_{y=\pi}$.



$$\psi = \frac{x}{2} - \frac{4}{\pi}\sum_{n=0}^{\infty} \frac{\sinh(2n+1)x \;\cos(2n+1)y}{\sinh(2n+1)\pi \;(2n+1)^2}$$

$$+ \frac{8}{\pi}\sum_{n=0}^{\infty} \frac{\sin(2n+1)x \;\cosh(2n+1)y}{(2n+1)^4 \;\sinh(2n+1)\pi}$$

(Still working on top edge)

$$\left.\frac{\partial\psi}{\partial y}\right|_{y=\pi} = (0)+(0)+\frac{8}{\pi}\sum_{n=0}^{\infty} \frac{\sin(2n+1)x \;(2n+1)\sinh(2n+1)\overset{\pi}{\cancel{y}}}{(2n+1)^4 \;\sinh(2n+1)\pi}$$

is this $x(\pi-x)$?

Write as $\sum_{n=0}^{\infty} \square_n \sin(2n+1)x \overset{?}{=} x(\pi-x)$

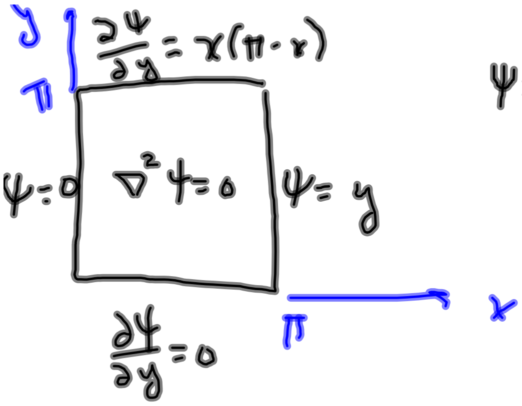
Sine series

$$\square_n = \frac{\int_0^{\pi} x(\pi-x)\sin(2n+1)x}{\int_0^{\pi} \sin^2(2n+1)x\,dx}$$

This turns to be $\frac{8}{\pi}\frac{1}{(2n+1)^3}$!!!

~~#5~~ #4 checks.

Not term-by-term – you have to sum it up.



$$\psi = \frac{x}{2} - \frac{4}{\pi}\sum_{n=0}^{\infty} \frac{\sinh(2n+1)x \;\; \cos(2n+1)y}{\sinh(2n+1)\pi \;\; (2n+1)^2}$$

$$+ \frac{8}{\pi}\sum_{n=0}^{\infty} \frac{\sin(2n+1)x \;\; \cosh(2n+1)y}{(2n+1)^4 \;\; \sinh(2n+1)\pi}$$

Eqn #5: the right edge.

$$y = \psi(\overset{\pi}{\cancel{x}}, y) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=0}^{\infty} \frac{\cos(2n+1)y}{(2n+1)^2} + \sum 0$$

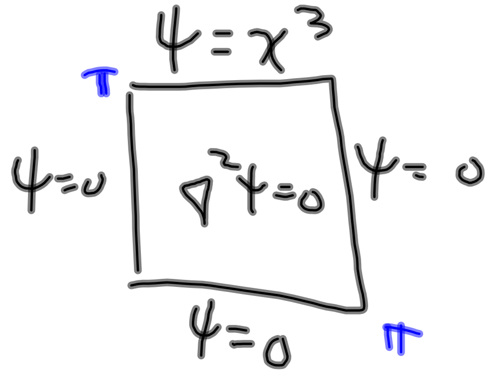
suspect this is a cos series for

$$\square + \sum_{n=0}^{\infty} \square_n \cos(2n+1)y = y$$

$$\square_n = \frac{\int_0^{\pi} y \cos(2n+1)y \, dy}{\int_0^{\pi} \cos^2(2n+1)y \, dy} = -\frac{4}{\pi}\frac{1}{(2n+1)^2}$$

$\pi/2$ or

In-class practice:
(#3 on p. 273)



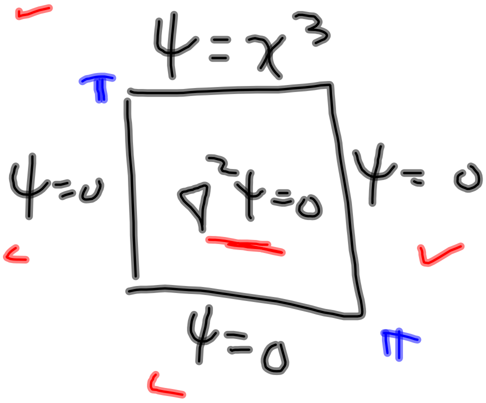
Check this formula:

$$\psi = \sum_{n=1}^{\infty} a_n \sin nx \sinh ny$$

$$a_n = \frac{\int_0^{\pi} x^3 \sin nx \, dx}{\int_0^{\pi} \sin^2 nx \, dx \ \sinh n\pi}$$

In-class practice:

(#3 on p.273)

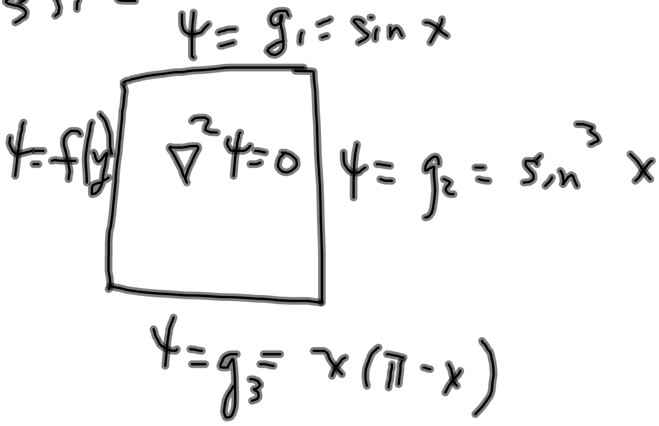


Check this formula:

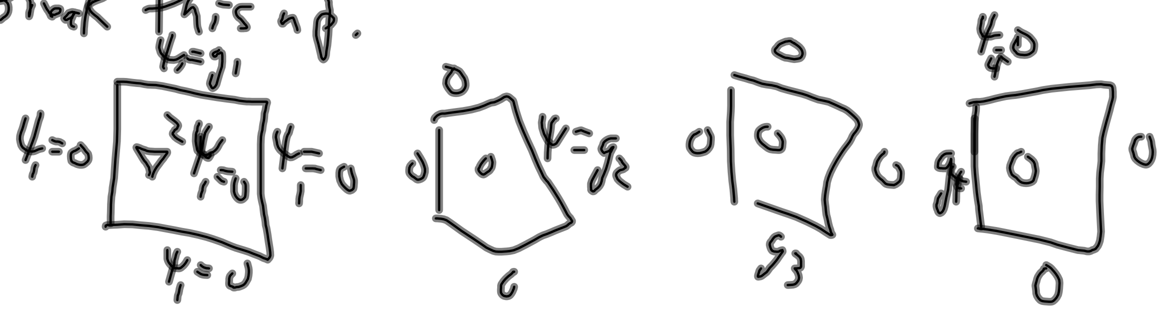
$$\psi = \sum_{n=1}^{\infty} a_n \overbrace{\sin nx \ \sinh ny}^{\nabla^2 = 0}$$

$$a_n = \frac{\int_0^{\pi} x^3 \sin nx \, dx}{\int_0^{\pi} \sin^2 nx \, dx \ \sinh n\pi}$$

§5.2

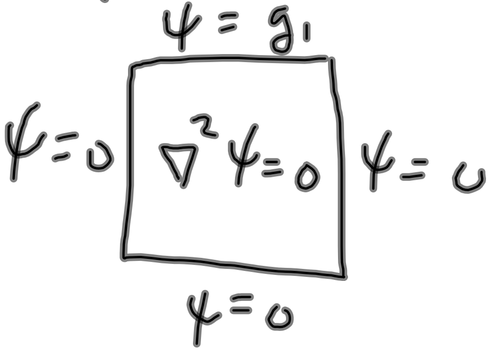


Break this up.



$$\psi = \psi_1 + \psi_2 + \psi_3 + \psi_4$$

Subproblem # 1 $\quad \psi(x,y) = X(x)\,Y(y)$ 1c



$X(x)$	$Y(y)$
$a_1 \cos kx + a_2 \sin kx$	$b_1 \cosh ky + b_2 \sinh ky$
$a_1 + a_2 x$	$b_1 + b_2 y$
$a_1 \cosh kx + a_2 \sinh kx$	$b_1 \cos ky + b_2 \sin ky$

for any a, b, k these solve the PDE.

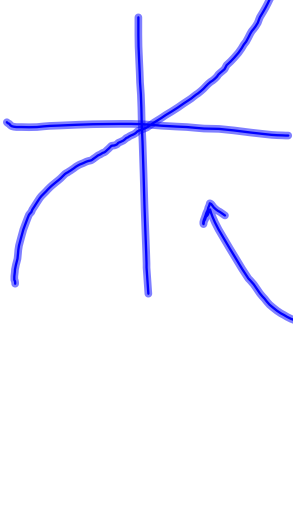
We will take combinations.

To force $\psi = 0$ on left:

$X(0) = 0 \qquad a_2 \sin kx$ or $a_2 x$ or $a_2 \sinh kx$

To force $\psi = 0$ at $x = \pi$

$X(\pi) = 0 \quad$ not by setting $a_2 = 0$ (get nothing)



? $\sin k\pi = 0 \qquad K = \cancel{0}, 1, 2, 3, 4, \cancel{4.5}, 5$ no help

$K = n$

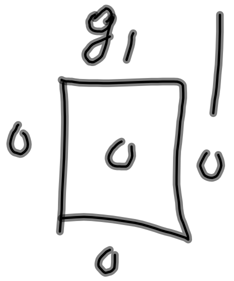
? $(x)_\pi = \pi = 0$ impossible

? $\sinh k\pi = 0$ '' also

So: the possibilities are

$$X(x) = \sin nx$$

This goes with $b_1 \cosh ny + b_2 \sinh ny$



need $\psi = 0$ at $y = 0$ Throw out the cosh.

$$\psi = \overset{a_2 b_2}{\boxed{}} \sin nx \sinh ny$$

To get g_1 on the top

$$\psi_{\text{general solution}} = \sum_{n=1}^{\infty} c_n \sin nx \sinh ny$$

I need

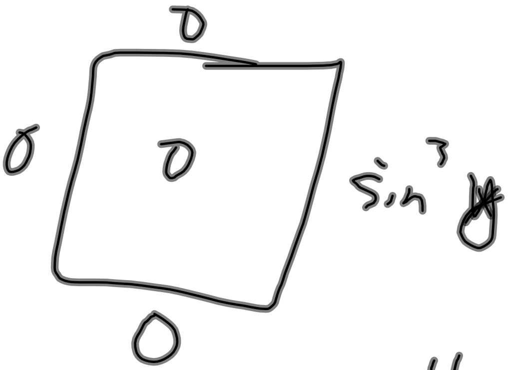
$$g_1(x) = \sum_{n=1}^{\infty} c_n \sin nx \sinh n\pi$$

$$\text{coef} = \frac{\int_0^{\pi} g_1(x) \sin nx \, dx}{\int_0^{\pi} \sin^2 nx \, dx} = c_n \sinh n\pi$$

$$c_n = \frac{\int g \sin}{\int \sin^2} \frac{1}{\sinh n\pi}$$

If $g_1 = \sin x$

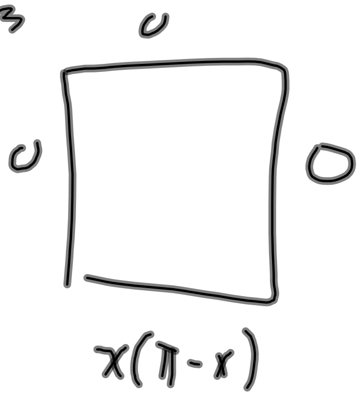
Sinh problem ?



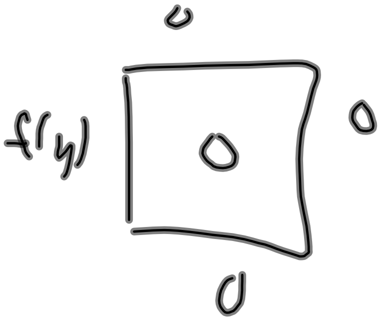
$$\psi = \sum_{n=1}^{\infty} c_n \sin ny \sinh nx$$

$$c_n = \frac{\int \sin^3 y \sin ny}{\int \sin^2 ny} \frac{1}{\sinh n\pi}$$

#3

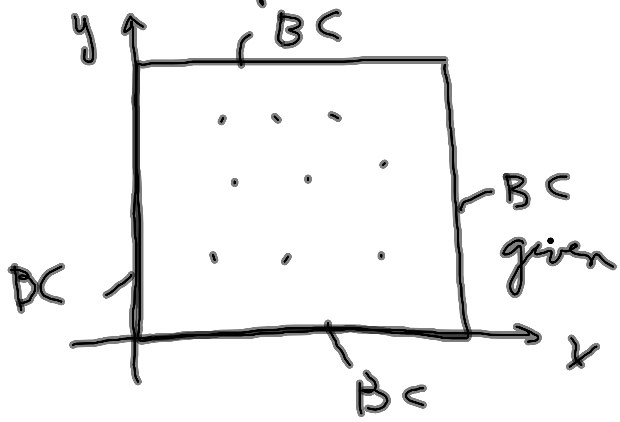


$$\psi = c_n \sin nx \quad \sinh n(\pi - y)$$



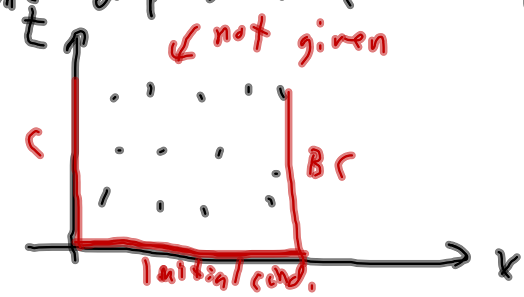
$$\psi = \sum c_n \sin ny \quad \cancel{n} \sinh n(\pi - x)$$

Time-independent (Static) problems



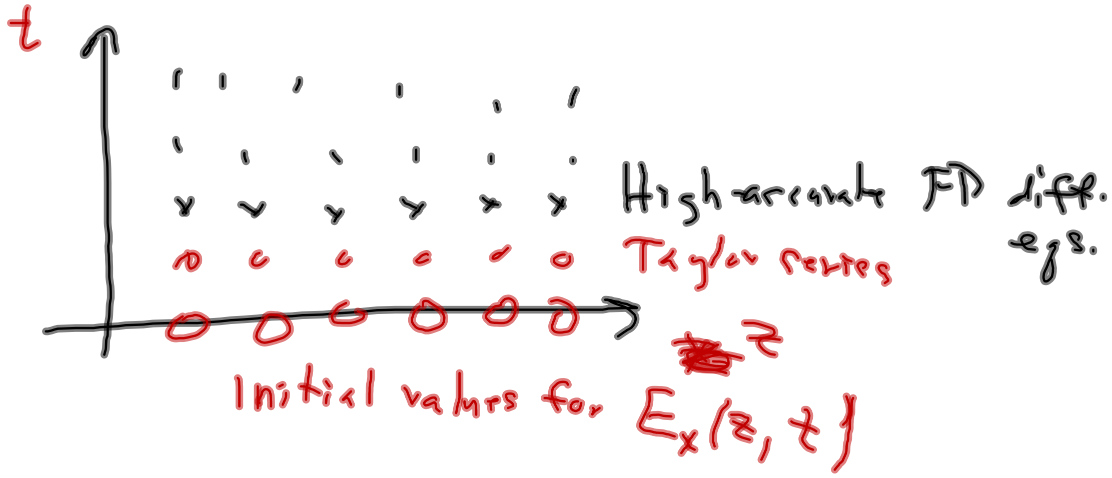
Keep updating interior values.

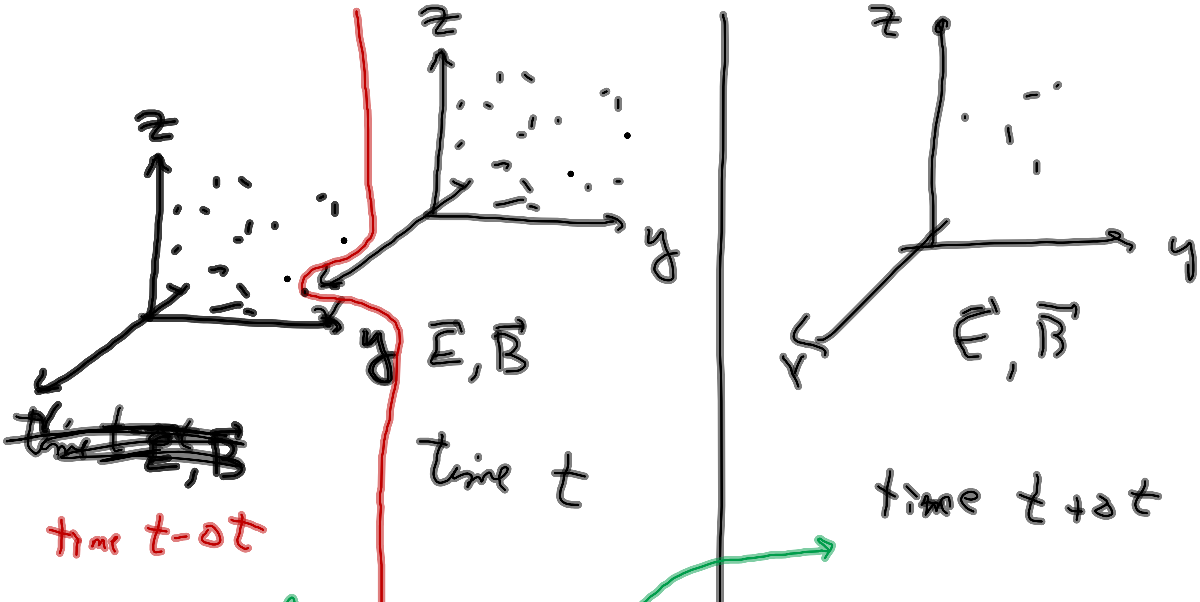
Time-dependent (wave eq.)



Directly compute
$V(\Delta t)$ in terms of $V(0)$,
$V(2\Delta t)$ " " " $V(\Delta t)$,
...... march directly in time.

Paul Garabedian


t
z
High-accurate FD diff. eqs.
Taylor series
Initial values for $E_x(z, t)$



$$\frac{\partial \vec{E}}{\partial t} = \left\{ E(t+\Delta t) - E(t-\Delta t) \right\} \cdot \frac{1}{2\Delta t}$$

$$\frac{\partial \vec{E}}{\partial t} = \nabla \times \vec{B}$$

Yee

$\frac{\partial B_y}{\partial x}$ rel. at time t

use centered differences as well.

Leapfrog F D T D scheme

inite ifference ime omain

developed by Yee, Taflove.

$$\frac{\partial \vec{E}}{\partial t} = -\nabla \times \vec{B} \qquad \frac{\partial \overset{\vec{B}/\mu}{\overset{\shortparallel}{\vec{H}}}}{\partial t} = \nabla \times \overset{\epsilon \vec{E}}{\overset{\shortparallel}{\vec{D}}}$$

What about the other Maxwell eqs ?

$$\vec{\nabla} \cdot \vec{D} = \rho \qquad \vec{\nabla} \cdot \vec{B} = 0 \qquad ?$$

http://www.amiright.com/parody/60s/thebeatles931.shtml

I was quizzical, studied quantum physical
Science at my school
Late nights all alone with my physics boo-oo-oo-ook
As I study here, to become an engineer
Friends call on the phone
Want me to come out, but I tell them no-oh-oh-oh
And as I say goodbye to my friends
I know what I must do

Bang, bang, Maxwell's four equations
I shove into my head
Bang, bang, Maxwell's four equations
My social life is dead

Back in class again hoping just to pass again
We all take our seats
Listening to what the professor say-eh-eh-ehs
Look it says right here "stick a Tesla in your ear"
Writing on the desk
Time to take a test to see what I know-oh-oh-oh
And as I sit here ready to start
I hope I've learned enough

Bang, bang, Maxwell's four equations
I pull out of my head
Bang, bang, Maxwell's four equations
I'm hanging by a thread

B-field density magnetic propensity
Maxwell stands with Gauss
Faraday the law of induction, oh-oh-oh-oh
Then there's Ampere's law, now my nerves are getting raw
All these I must learn (These things you must learn)
A good grade I must earn for my mind to grow-oh-oh-oh
And if I ever figure them out
It will not be too soon

Bang, bang, Maxwell's four equations
I know inside my head
Bang, bang, Maxwell's four equations
I'll keep them 'til I'm dead

$\cdot\ \mathbf{D} = \rho$ (Gauss' law of electricity)
$\cdot\ \mathbf{B} = 0$ (Gauss' law of magnetism)
x $\mathbf{E} = -(\partial \mathbf{B}/\partial t)$ (Faraday's law of induction)
x $\mathbf{H} = \mathbf{J} + \partial \mathbf{D}/\partial t$ (Ampère's law) with Maxwell displacement current

Fact: If $\nabla \cdot \vec{D} = \rho$ at $t=0$

$\nabla \cdot \vec{B} = 0$ at $t = 0$

they will be preserved through time !!!

Proof. I have inserted corrections to the lecture.

$$\frac{\partial \vec{D}}{\partial t} = \nabla \times \vec{H} - \vec{j}$$

Take div

$$\vec{\nabla} \cdot \frac{\partial \vec{D}}{\partial t} = \frac{\partial}{\partial t} \vec{\nabla} \cdot \vec{D} = \vec{\nabla} \cdot (\vec{\nabla} \times \vec{H}) - \vec{\nabla} \cdot \vec{j}$$

div curl $\equiv 0$

So $\frac{\partial \nabla \cdot \vec{D}}{\partial t} = -\nabla \cdot \vec{j}$

$\frac{\partial \rho}{\partial t} = -\nabla \cdot \vec{j}$ $\qquad = \frac{\partial \rho}{\partial t}$

$$\frac{\partial [\nabla \cdot \vec{D} - \rho]}{\partial t} = 0 \Rightarrow \vec{\nabla} \cdot D - \rho \text{ is constant}$$

started at 0,
always 0.

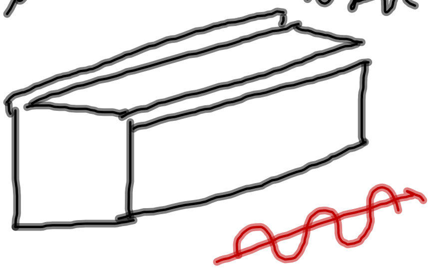
$$\frac{\partial \vec{B}}{\partial t} = -\nabla \times \vec{E}$$

$$\frac{\partial\, \nabla\cdot\vec{B}}{\partial t} = -\nabla\cdot\left(\nabla\times\vec{E}\right) \equiv 0 \quad \text{div curl} = 0$$

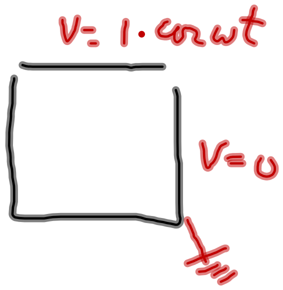
so $\nabla\cdot B$ = constant, initially zero,

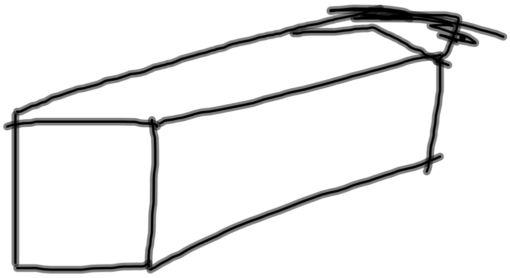
always zero.

Next time: wave guides



front view



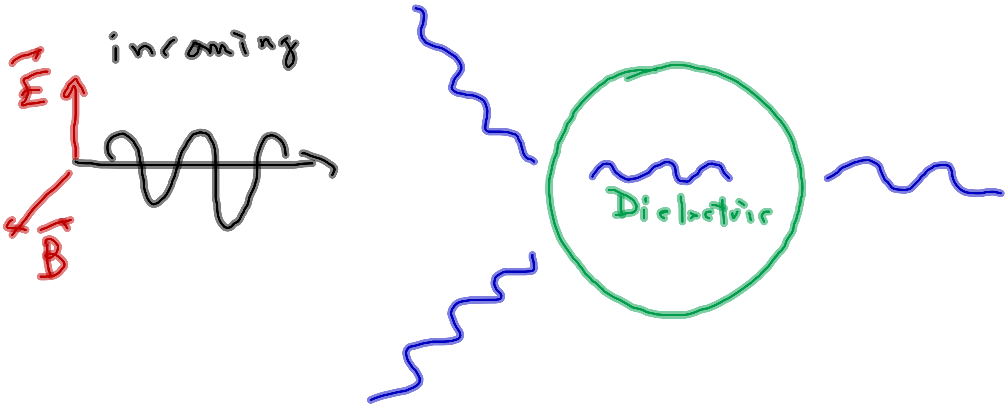


Front view

electrons.

<u>will</u> propagate if ω is fast enough.

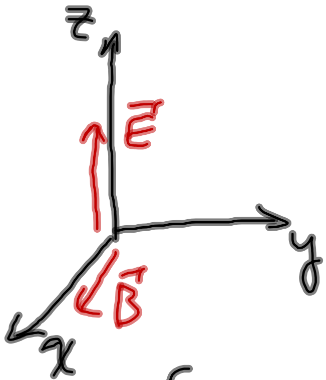
EM scattering off of a dielectric sphere.

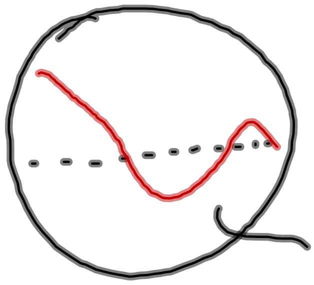


Exact solution was worked out 1903 by Mie.

FDTDchanged.m

max $|E_z|/|E_{inc}|$ over a $\frac{1}{2}$-period.





index of refraction = 2

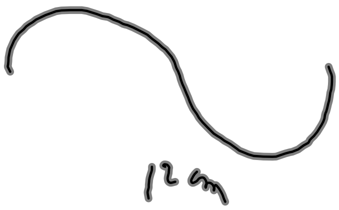
freq. 2.5 GHz

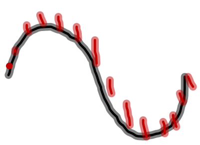
$c_{vac} = 3\times10^8$ m/s

$$\text{wavelength}_{vac} = \frac{3\times10^8}{2.5\times10^9} = .12 \text{ m} = 12 \text{ cm}.$$

$$\text{wavelength}_{diel} = 6 \text{ cm}.$$

vac.





$$\Delta x = \frac{\lambda_{diel.}}{20} = .003 \text{ m}$$

Stability

$$\left|\frac{\Delta t}{\Delta x}\right| < \frac{1}{c} \qquad |\Delta t| \leq \frac{\Delta x}{c}$$

$\Delta x = .003\,m$ $\quad$ Freq $= 2.5 \times 10^9$

$$\Delta t = \left[\frac{\Delta x}{c}\right]\frac{1}{2} = \frac{1}{2}\,\frac{3\times10^{-3}}{3\times10^{8}} = 5\times10^{-12}\text{ sec.}$$

$$\text{Period} = \frac{1}{\text{Freq.}} = \frac{1}{2.5\times10^{9}} = 4\times10^{-10}\text{ sec.}$$

6 periods $= 24\times10^{-10}$ sec. $= 2.4\times10^{-9}$ sec.

<u>stability</u>: $|\Delta t| \le f(\Delta x)$

wave $|\Delta t| < \frac{\Delta x}{c}$

heat $|\Delta t| \le (\Delta x)^2/\text{constant}$

Reason: "von Neumann"

"Courant/Friedrichs/Levy" (CFL)

The potential Theorems of Vector Analysis and their advantages.

"There is a scalar potential for the vector $\vec{F}(x,y,z) = -\nabla\phi(x,y,z)$"

if and only if

$$\nabla \times \vec{F} = \vec{0}$$

Savings $\vec{F}$ has 3 unknown components, ϕ has only one.

"There is a vector potential for $\vec{F} = \nabla \times \vec{G}$"

if and only if

$$\vec{\nabla} \cdot \vec{F} = 0 .$$

Savings Both $\vec{F}$ and $\vec{G}$ have 3 unknown components, but you can impose an extra condition on $\vec{G}$.

(Called "Gauge" condition)

Separation of variables: Try $\psi(x, y, z, t) = \psi_{2-D}(x, y)\, Z(z)\, T(t)$

Look for "easy" solutions.

Try to express general solutions as combinations of easy solutions.

Time:

$$T(t) = e^{j\omega t}$$

Waveguide

Case 1: the top is insulated

Then you can chose the top at 1 volt + ground the 3 sides at DC.

Now oscillate the voltage ±1 volt, you get a wave moving down the guide. At any frequency, no distortion, TEM, simply take the DC solution and slap on a propagation factor $e^{i(\omega t - kz)}$

$K = \omega \sqrt{\epsilon \mu}$. (Speed of light.)

Case 2. The top is shorted to the sides.

No DC – electrons short out any voltage that you try to apply. But if you switch voltages ±1 faster than the electrons can short it out, you get a wave. These waves have cutoff frequencies, distortion.

Mx eqs with factor $e^{j\omega t}$

1. $\nabla \cdot \vec{D} = \vec{\nabla} \cdot \epsilon \vec{E} = \epsilon \, \vec{\nabla} \cdot \vec{E} = \rho$

2. $\nabla \times \vec{E} = -j\omega \vec{B}$

3. $\vec{\nabla} \cdot \vec{B} = 0$

4. $\vec{\nabla} \times \vec{H} = j\omega \vec{D} + \vec{J}$

Mx eqs.

1. $\nabla \cdot \vec{D} = \vec{\nabla} \cdot \epsilon \vec{E} = \epsilon \vec{\nabla} \cdot \vec{E} = \rho$

2. $\nabla \times \vec{E} = -j\omega \vec{B}$

3. $\vec{\nabla} \cdot \vec{B} = 0$

4. $\vec{\nabla} \times \vec{H} = j\omega \vec{D} + \vec{J}$

$$\frac{\partial \rho}{\partial t} = -\vec{\nabla} \cdot \vec{j} = j\omega\rho$$

This at $4 \Rightarrow 1$ so forget 1

Mx eqs.

1. $\nabla\cdot\vec{D} = \vec{\nabla}\cdot\epsilon\vec{E} = \epsilon\,\vec{\nabla}\cdot\vec{E} = \rho$

2. $\nabla\times\vec{E} = -j\omega\vec{B}$

3. $\vec{\nabla}\cdot\vec{B} = 0$

4. $\vec{\nabla}\times\vec{H} = j\omega\vec{D} + \vec{J}$

4 ⇒ 1 so forget 1

3 ⇔ $\vec{B} = \vec{\nabla}\times\vec{A}$ use $\vec{A}$, forget 3.

M's eqs.

1. $\nabla\cdot\vec{D} = \nabla\cdot\epsilon\vec{E} = \epsilon\,\nabla\cdot\vec{E} = \rho$

2. $\nabla\times\vec{E} = -j\omega\vec{B}$

3. $\nabla\cdot\vec{B} = 0$

4. $\nabla\times\vec{H} = j\omega\vec{D} + \vec{J}$

4 $\Rightarrow$ 1 so forget 1

3 $\Rightarrow$ $\vec{B} = \underbrace{\nabla\times\vec{A}}$ use $\vec{A}$, forget 3.

2. $\nabla\times\vec{E} + j\omega\vec{B} = 0 = \nabla\times[\underbrace{\vec{E} + j\omega\vec{A}}_{=-\nabla\phi}]$

use $\vec{E} = -j\omega\vec{A} - \nabla\phi$ and forget # 2.

So we only have solve # 4 written in terms of $\vec{A}$ & ϕ.

(get $\vec{E}$, $\vec{B}$ from $\vec{A}$, ϕ after we calculate the latter.

Mx eqs. with $\rho = \vec{J} = 0$

1. $\nabla \cdot \vec{D} = \vec{\nabla} \cdot \epsilon \vec{E} = \epsilon \vec{\nabla} \cdot \vec{E} = \rho = 0$
2. $\nabla \times \vec{E} = -j\omega \vec{B}$
3. $\vec{\nabla} \cdot \vec{B} = 0$
4. $\vec{\nabla} \times \vec{H} = j\omega \vec{D} + \cancel{\vec{J}}^{\,0}$ $\quad \nabla \times \vec{B} = j\omega\epsilon\mu \vec{E}$

$4 \Rightarrow 1$ so forget 1

$3 \Leftrightarrow \vec{B} = \vec{\nabla} \times \vec{A}$ use $\vec{A}$, forget 3.

2. $\nabla \times \vec{E} + j\omega \vec{B} = 0 = \vec{\nabla} \times [\underbrace{\vec{E} + j\omega \vec{A}}_{= -\nabla \phi}]$

so use $\vec{E} = -j\omega \vec{A} - \nabla \phi$ and forget # 2.

So we only have solve # 4, written in terms of $\vec{A}$ & ϕ.

(Get $\vec{E}, \vec{B}$ from $\vec{A}, \phi$ after we calculate the latter)

LHS $\nabla \times B = \vec{\nabla} \times (\vec{\nabla} \times A) = -\nabla^2 \vec{A} + \nabla(\vec{\nabla} \cdot \vec{A})$

RHS $j\omega\epsilon\mu \vec{E} = j\omega\epsilon\mu(-j\omega\vec{A} - \nabla\phi)$

LHS = RHS

$$-\nabla^2 \vec{A} - \omega^2 \epsilon\mu \vec{A} = \vec{\nabla}\underbrace{\left[-\vec{\nabla} \cdot \vec{A} - j\omega\epsilon\mu\phi\right]}_{0}$$

Exploit the flexibility in the vector potential:

set $\phi = -\frac{\vec{\nabla} \cdot \vec{A}}{j\omega\epsilon\mu}$

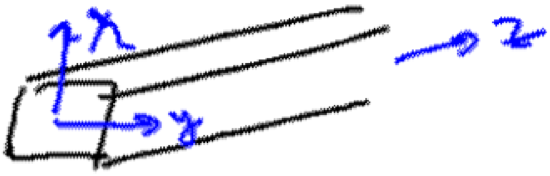
Gauge condition"

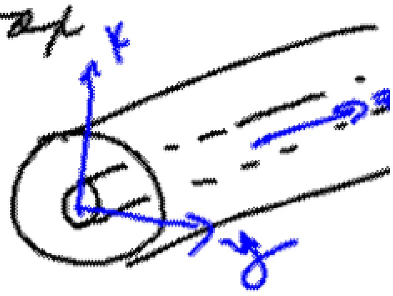
$$\boxed{\nabla^2 \Psi + \omega^2 \epsilon\mu \Psi = 0}$$

Helmholtz Equation

$\nabla^2 \vec{A} + \omega^2 \epsilon\mu \vec{A} = 0$ 3 uncoupled eqs : A_x, A_y, A_z

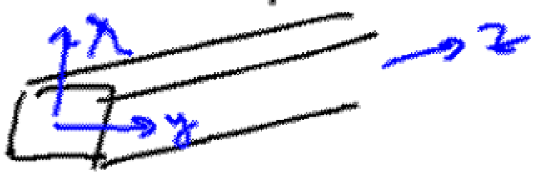
Waveguide or coax

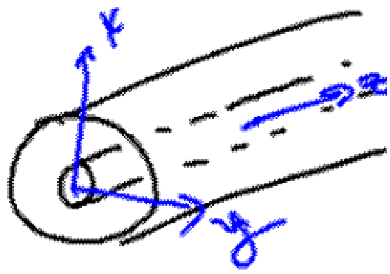




$\nabla^2 \vec{A} + \omega^2 \epsilon\mu \vec{A} = 0$ 3 uncoupled eqs: A_x, A_y, A_z

Waveguide or coax

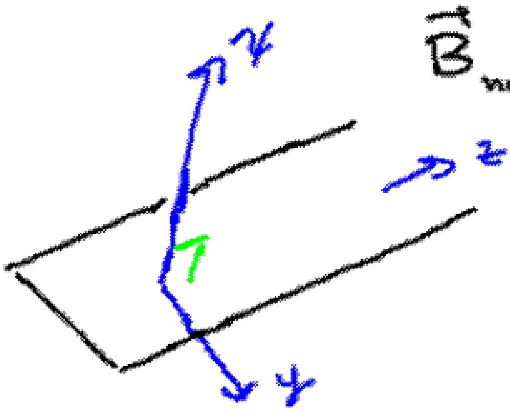




Boundary conditions: $\vec{E}_{tan}\Big|_{wall} = 0$

$\vec{B}_{normal}\Big|_{wall} = 0$

Bottom Wall:



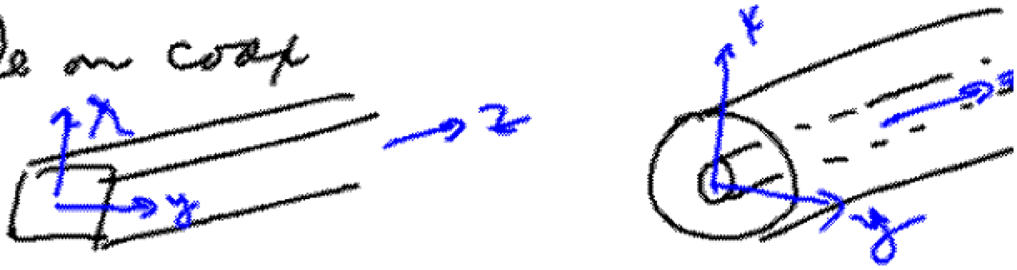
at wall,

$E_y = E_z = 0$

$B_x = 0$

$\nabla^2 \vec{A} + \omega^2 \epsilon\mu \vec{A} = 0$ 3 uncoupled eqs: A_x, A_y, A_z

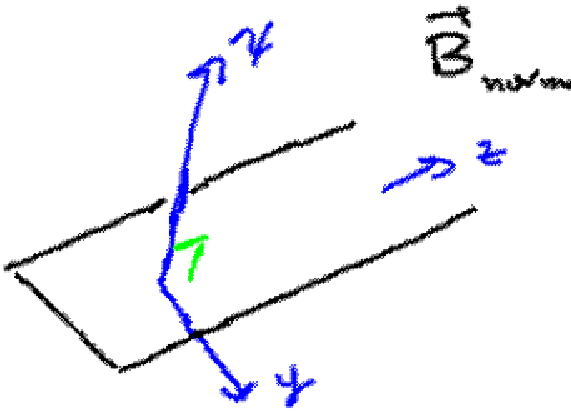
Waveguide or coax



Boundary conditions: $\vec{E}_{tan}\Big|_{wall} = 0$

$\vec{B}_{normal}\Big|_{wall} = 0$

Bottom Wall:



at wall,
$E_y = E_z = 0$
$B_x = 0$

Helmholtz Trick: take $\vec{A} = \psi \vec{k}$

~~Clea~~ Claim: this is a Transverse Magnetic wave;

$$\vec{B} = \vec{\nabla} \times \vec{A} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ 0 & 0 & \psi \end{vmatrix} = \vec{i}\,\partial_y \psi + \vec{j}\,(-\partial_x \psi) + \vec{k}\,(0)$$

We have already imposed 1 condition on vector potential, which is OK. This new condition is restrictive - we will not get the most general solution.

$$\nabla^2 \psi + \omega^2 \epsilon\mu \psi = 0$$ Helmholtz Equation

Try separating $\psi = \psi_t(x,y)\, Z(z)$

$$BC \Rightarrow B_n = 0 = B_x = \frac{\partial}{\partial y}\psi \quad \text{on the wall,}$$

$$= \frac{\partial}{\partial y}\psi_t(x,y)\, Z(z)$$

$$\text{so} \quad \psi_t = \text{constant on the wall.}$$

$$\boxed{\psi(x,y,z) = \underbrace{\psi_t(x,y)}_{\substack{\parallel \\ \text{constant on wall}}}\, Z(z)}$$

$$\cancel{\nabla^2_{2D}\psi_t(x,y) + \omega^2 \epsilon\mu\, \psi_t(x,y) = 0}, \quad \psi_t = \text{constant on each wall}$$

$$\nabla^2_{2D}\psi_t = \lambda\, \psi_t \, ,$$

2 dimensional Helmholtz Equation

So far: $\psi(x,y,z) = \psi_t(x,y)\, Z(z)$

2 dimensional Helmholtz Equation $\nabla^2 \psi_t = \lambda_1 \psi_t$ $\qquad \psi_t = \text{const.} \quad \left(\vec{B}_{normal} = 0\right)$

Now Enforce $E_x^{tan} = E_y = 0$ on walls.

$$\vec{E} = -j\omega \vec{A} - \vec{\nabla}\Big(\underbrace{-\frac{1}{j\omega\epsilon\mu}\vec{\nabla}\cdot\vec{A}}_{\phi}\Big) \qquad \vec{A} = \psi \vec{k}$$

$$\frac{\partial}{\partial z}\psi = \frac{\partial}{\partial z}\psi_t(x,y)\, Z(z)$$

use form $Z(z) = e^{\pm\sqrt{\lambda_2}\, z}$

instead of $\cos$/$\sin$ $\sqrt{-\lambda_2}\, z$

$\psi_t Z$

$$= -j\omega \psi \vec{k} + \frac{1}{j\omega\epsilon\mu}\vec{\nabla}\Big(\underbrace{\pm\sqrt{\lambda_2}\,\psi_t Z}\Big)$$

$=$

$$\pm\sqrt{\lambda_2}\, Z\, \vec{\nabla}_{2D}\psi_t + \left(\pm\sqrt{\lambda_2}\right)^2 \psi_t Z \vec{k}$$

$$\vec{E} = \vec{k}\left[\psi_t e^{\pm\sqrt{\lambda_2} z}\left(j\omega + \frac{1}{j\omega\epsilon\mu}\left[\pm\sqrt{\lambda_2}\right]^2\right)\right]$$

$$\pm\frac{\sqrt{\lambda_2}}{j\omega\epsilon\mu} e^{\pm\sqrt{\lambda_2}\, z}\, \vec{\nabla}_{2D}\underbrace{\psi_t(x,y)}$$

Need to enforce $\vec{E}_{tan} = 0$

Tang. comp. of this is zero on wall; since ψ_t = const on wall, $\vec{\nabla}_{2D}\psi_t$ is $\perp$ to the wall.

$$* \quad \psi_{t(x,y)} e^{\pm\sqrt{\lambda_2} z}\left(j\omega + \frac{1}{j\omega\epsilon\mu}\left[\pm\sqrt{\lambda_2}\right]^2\right)$$

need to make $* = 0$ on wall, since E_z IS a a tangential component.

Easy way $\psi_t = 0$ on all walls

(What's left of the 3 dimensional Helmholtz Equati

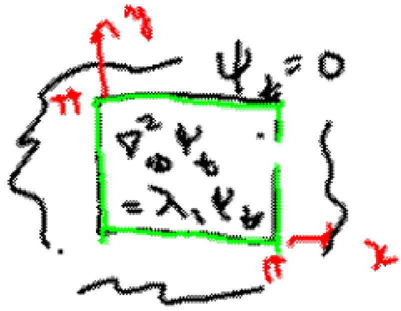
$$\lambda_1 + \lambda_2 + \omega^2\epsilon\mu = 0$$

2 dimensional Helmholtz Equation

$$\nabla^2_{2D}\psi_t = \lambda_1 \psi_t$$

$\psi_t = 0$ on wall

Single conductor



$\psi_t = \sin x \sin y$ works!

$$\nabla^2_{2D}\psi_t = -\sin x \sin y - \sin x \sin y$$

$$= -2\psi_t . \qquad \lambda_1 = -2$$

* $E_z = \psi_t(x,y)\, e^{\pm\sqrt{\lambda_2}z}\left(j\omega + \frac{1}{j\omega\epsilon\mu}\left[\pm\sqrt{\lambda_2}\right]^2\right)$ must equal zero on wall

Hard way $\quad -j\omega + \frac{1}{j\omega\epsilon\mu}\left[\pm\sqrt{\lambda_2}\right]^2 = 0$

$$\Rightarrow \lambda_2 = -\omega^2\epsilon\mu.$$

note: E_z will be zero everywhere: this field is also T_v Electric.

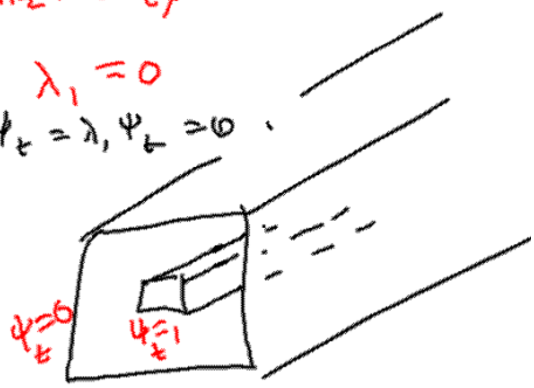
~~Ans~~ Don't care what constant ψ_t is on the wall.

$$\lambda_1 + \lambda_2 + \omega^2\epsilon\mu = 0$$

$$\Rightarrow \lambda_1 = 0$$

$$\nabla^2_{xy}\psi_t = \lambda_1\psi_t = 0.$$

GREAT!



Solve $\nabla^2_{xy}\psi_t = 0$ with these BC's,

$\Rightarrow \psi_t(x,y)$.

Attach $Z(z) = e^{\pm\sqrt{\lambda_2}z}$

$$= e^{\pm j\omega\sqrt{\epsilon\mu}z}$$

$$= \begin{matrix}\cos\\ \sin\end{matrix}\left(\omega\sqrt{\epsilon\mu}\, z\right)$$

Works for any ω down to 0 (D.C)

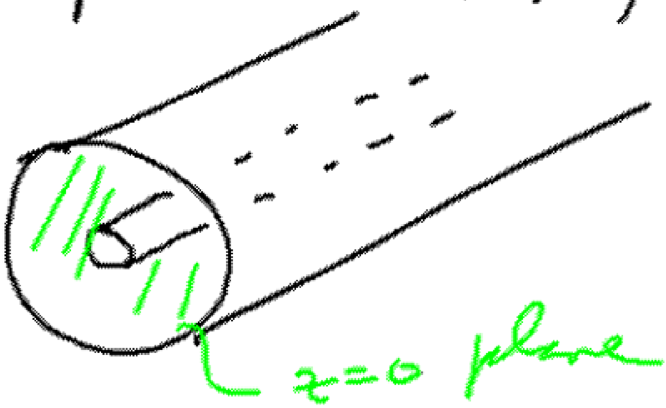
$$\psi = \psi_t(x,y)\cos\left[\omega\sqrt{\epsilon\mu}\, z - \omega t\right]$$

Non-dispersive traveling wave, going down the waveguide at speed

wave front $\quad \omega\sqrt{\epsilon\mu}\, z - \omega t = \text{constant} = 0$

$$z/t = \frac{1}{\sqrt{\epsilon\mu}} = \text{speed}$$

$E_z = 0$ $B_z = 0$ for these solns; They are TEM



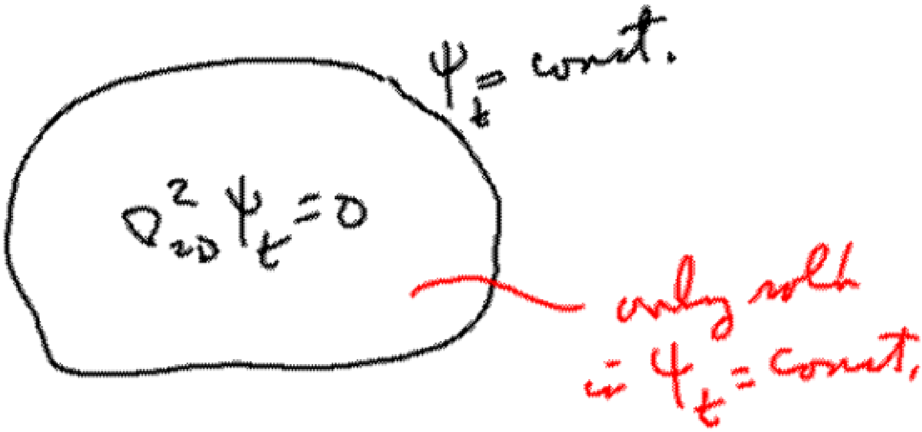
In z=0 plane,

To get the field at other z, multiply this ~~plot~~ pattern by $\cos[\omega(\sqrt{\epsilon\mu}\, z - t)]$ or sin, ...

If $\omega = 0$, $\cos(0) = 1$; static, DC, z-invariant solution.

~~With this strategy, need to solve~~

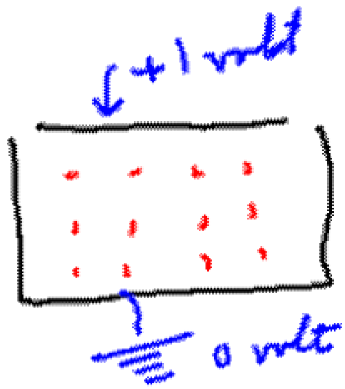
With a single conductor in this model you must solve



So This (TEM) sol'n is available only for a multiple conductor.

Thus waveguides have no TEM wave.

Case 1 math:



Solve the DC potential:

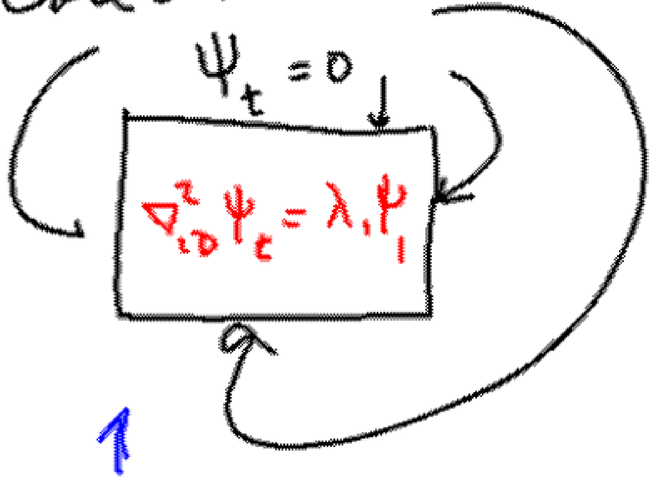
$\nabla^2 \phi = 0$, $\phi = \begin{cases} 1 & \text{on top} \\ 0 & \text{on bottom} \end{cases}$

$\phi = \phi(x, y)$

Use finite difference, Laplace molecule, relaxation.

Attach $e^{i(\omega t - kz)}$ $k = \omega\sqrt{\epsilon\mu}$ to $\phi(x, y)$.

Case 2 math.



Solve

$$\boxed{\nabla^2_{2\text{-}D}\,\psi_t(x,y) = \lambda_1\,\psi_t(x,y)}$$

Slap on factor $e^{j(\omega t - Kz)}$

$$K = \sqrt{\omega^2 \epsilon\mu + \lambda_1}$$

This problem only has nontrivial solution for discrete set of eigenvalues λ_1. Typically $0 > \lambda_1 > \lambda_2 \ldots$

The Cutoff Frequency: Suppose $\lambda_1 = -10$.

$$K = \sqrt{\omega^2 \epsilon\mu - 10}$$

If ω is too low ($\omega^2 \epsilon\mu - 10 < 0$), K is imaginary, e^{-jKz} is $e^{-j^2 \# z}$ = exponential growth or decay.

Lowest Cutoff freq. is $\omega^2 \epsilon\mu + \lambda_1 > 0$

$\cancel{\omega = \sqrt{\frac{\lambda_1}{\epsilon_1}}}$

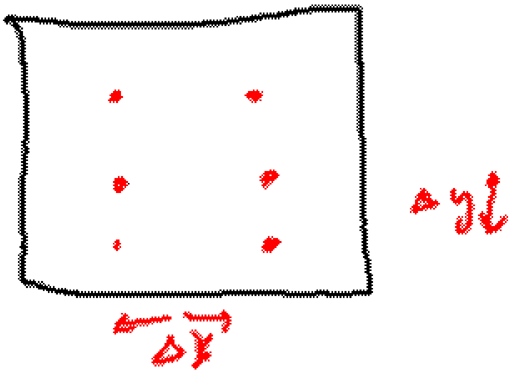
$$\omega = \sqrt{\frac{-\lambda_1}{\epsilon\mu}} \quad (\lambda_1 < 0)$$

How do you solve $\psi_t = 0$ numerically

$$\boxed{\nabla^2 \psi_t = \lambda \psi_t}$$

You don't know λ at the outset. Only privileged values give you a nontrivial solution.

1. Write finite difference equations for $\nabla^2 \psi_t = \lambda \psi_t$
2. Rewrite as a matrix system. It will look like $\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} \begin{bmatrix} \psi \\ \vdots \end{bmatrix} = \lambda \begin{bmatrix} \psi \\ \vdots \end{bmatrix}$.
3. Ask MATLAB for values of ψ & λ.



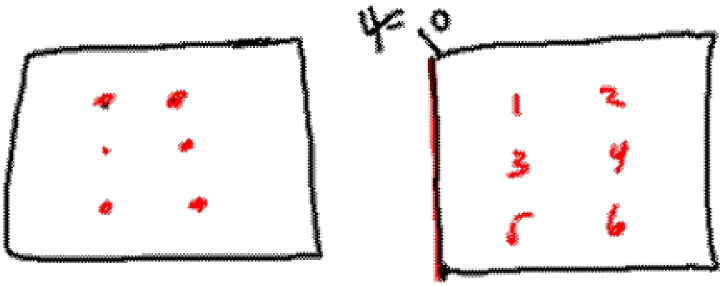
$$\nabla^2\psi = \frac{\psi_\uparrow + \psi_\downarrow + \psi_\leftarrow + \psi_\rightarrow - 4\psi_0}{\Delta^2}$$

From Sadiku's table

$$\frac{\partial^2\psi}{\partial x^2} = \frac{\psi_\rightarrow + \psi_\leftarrow - 2\psi_0}{\Delta x^2} \qquad \frac{\partial^2\psi}{\partial y^2} = ''$$

$$\nabla^2\psi \approx \frac{\psi_\rightarrow + \psi_\leftarrow - 4\psi_0 + \psi_\uparrow + \psi_\downarrow}{\Delta x^2} + O(\Delta x^2)$$

$\Delta x = \Delta y$.



$$\nabla^2\psi = \lambda\psi$$

$$\nabla^2\psi \approx \frac{\psi_\uparrow + \psi_\downarrow + \psi_\leftarrow + \psi_\rightarrow \ldots}{\Delta x^2}$$

At node #1 $\quad \dfrac{0+\psi_3+0+\psi_2-4\psi_1}{\Delta x^2} = \lambda\psi_1$ $\qquad$ #5 $\quad \dfrac{\psi_3+0+\psi_6+0-4\psi_5}{\Delta x^2} = \lambda\psi_5$

At node #2! $\quad \dfrac{0+\psi_4+0+\psi_1-4\psi_2}{\Delta x^2} = \lambda\psi_2$

#3: $\quad \dfrac{\psi_1+\psi_5+\psi_4+0-4\psi_3}{\Delta x^2} = \lambda\psi_3$ $\qquad$ #6 $\quad \dfrac{\psi_4+0+0+\psi_5-4\psi_6}{\Delta x^2} = \lambda\psi_6$

#4: $\quad \dfrac{\psi_2+\psi_6+0+\psi_3-4\psi_4}{\Delta x^2} = \lambda\psi_4$

$$\underbrace{\frac{1}{\Delta x^2}\begin{bmatrix} -4 & 1 & 1 & 0 & 0 & 0 \\ 1 & -4 & 0 & 1 & 0 & 0 \\ 1 & 0 & -4 & 1 & 1 & 0 \\ 0 & 1 & 1 & -4 & 0 & 1 \\ 0 & 0 & 1 & 0 & -4 & 1 \\ 0 & 0 & 0 & 1 & 1 & -4 \end{bmatrix}}_{[A]} \underbrace{\begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \\ \psi_5 \\ \psi_6 \end{bmatrix}}_{[\psi]} = \lambda \underbrace{\begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \\ \psi_5 \\ \psi_6 \end{bmatrix}}_{[\psi]}$$

The potential Theorems of Vector Analysis and their advantages.

"There is a scalar potential for the vector $\vec{F}(x,y,z) = -\nabla\phi(x,y,z)$"

if and only if

$$\nabla \times \vec{F} = \vec{0}$$

savings $\vec{F}$ has 3 unknown components, ϕ has only one.

"There is a vector potential for $\vec{F} = \nabla \times \vec{G}$"

if and only if

$$\vec{\nabla} \cdot \vec{F} = 0 .$$

Savings Both $\vec{F}$ and $\vec{G}$ have 3 unknown components, but you can impose an extra condition on $\vec{G}$.

(Called "Gauge" condition)

Separation of variables: Try $\psi(x, y, z, t) = \psi_{2-D}(x, y)\, Z(z)\, T(t)$

Look for "easy" solutions.

Try to express general solutions as combinations of easy solutions.

Time:

$$T(t) = e^{j\omega t}$$

Waveguide

Case 1: the top is insulated

Then you can charge the top at 1 volt + ground the 3 sides at DC.

Now oscillate the voltage ± 1 volt, you get a wave moving down the guide. At any frequency, no distortion, TEM, simply take the DC solution and slap on a propagation factor $e^{i(\omega t - kz)}$

$K = \omega \sqrt{\epsilon \mu}$. (Speed of light.)

Case 2. The top is shorted to the sides.
no DC - electrons short out any voltage that you try to apply. But if you switch voltage ± 1 faster than the electrons can short it out, you get a wave. These waves have cutoff frequencies, distortion.

Mx eqs with factor $e^{j\omega t}$

1. $\nabla \cdot \vec{D} = \vec{\nabla} \cdot \epsilon \vec{E} = \epsilon \, \vec{\nabla} \cdot \vec{E} = \rho$

2. $\nabla \times \vec{E} = -j\omega \vec{B}$

3. $\vec{\nabla} \cdot \vec{B} = 0$

4. $\vec{\nabla} \times \vec{H} = j\omega \vec{D} + \vec{J}$

Mx eqs.

1. $\nabla \cdot \vec{D} = \vec{\nabla} \cdot \epsilon \vec{E} = \epsilon \vec{\nabla} \cdot \vec{E} = \rho$

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3. $\vec{\nabla} \cdot \vec{B} = 0$

4. $\vec{\nabla} \times \vec{H} = j\omega \vec{D} + \vec{J}$

$$\frac{\partial \rho}{\partial t} = -\vec{\nabla} \cdot \vec{j} = j\omega\rho$$

This out $4 \Rightarrow 1$ so forget 1

Mx eqs.

1. $\nabla\cdot\vec{D} = \nabla\cdot\epsilon\vec{E} = \epsilon\,\nabla\cdot\vec{E} = \rho$

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$3 \Leftrightarrow \vec{B} = \nabla\times\vec{A}$ use $\vec{A}$, forget 3.

Mx's eqs.

1. $\nabla\cdot\vec{D} = \nabla\cdot\epsilon\vec{E} = \epsilon\,\nabla\cdot\vec{E} = \rho$

2. $\nabla\times\vec{E} = -j\omega\vec{B}$

3. $\nabla\cdot\vec{B} = 0$

4. $\nabla\times\vec{H} = j\omega\vec{D} + \vec{J}$

$4 \Rightarrow 1$ so forget 1

$3 \Rightarrow \vec{B} = \underbrace{\nabla\times\vec{A}}$ use $\vec{A}$, forget 3.

2. $\nabla\times\vec{E} + j\omega\vec{B} = 0 = \nabla\times\left[\underbrace{\vec{E} + j\omega\vec{A}}_{= -\nabla\phi}\right]$

Use $\vec{E} = -j\omega\vec{A} - \nabla\phi$ and forget # 2.

So we only have solve # 4, written in terms of $\vec{A}$ & ϕ.

(Get $\vec{E}, \vec{B}$ from $\vec{A}, \phi$ after we calculate the latter.

M's eqs. with $\rho = \vec{J} = 0$

1. $\nabla \cdot \vec{D} = \vec{\nabla} \cdot \epsilon \vec{E} = \epsilon\, \vec{\nabla} \cdot \vec{E} = \rho = 0$
2. $\nabla \times \vec{E} = -j\omega \vec{B}$
3. $\vec{\nabla} \cdot \vec{B} = 0$
4. $\vec{\nabla} \times \vec{H} = j\omega \vec{D} + \cancel{\vec{J}}^{\,0}$ $\qquad \nabla \times \vec{B} = j\omega\epsilon\mu \vec{E}$

$4 \Rightarrow 1$ so forget 1

$3 \Leftrightarrow \vec{B} = \underbrace{\vec{\nabla} \times \vec{A}}$ use $\vec{A}$, forget 3.

2. $\nabla \times \vec{E} + j\omega \vec{B} = 0 = \vec{\nabla} \times [\underbrace{\vec{E} + j\omega \vec{A}}_{= -\nabla\phi}]$

use $\vec{E} = -j\omega \vec{A} - \nabla\phi$ and forget # 2.

So we only have solve # 4, written in terms of $\vec{A}$ & ϕ.
(get $\vec{E}, \vec{B}$ from $\vec{A}, \phi$ after we calculate the latter)

LHS $\nabla \times B = \vec{\nabla} \times (\vec{\nabla} \times A) = -\nabla^2 \vec{A} + \nabla(\vec{\nabla} \cdot \vec{A})$

RHS $j\omega\epsilon\mu \vec{E} = j\omega\epsilon\mu(-j\omega \vec{A} - \nabla\phi)$

LHS RHS

LHS = RHS

$-\nabla^2 \vec{A} - \omega^2 \epsilon\mu \vec{A} = \vec{\nabla}\underbrace{[-\vec{\nabla} \cdot \vec{A} - j\omega\epsilon\mu\phi]}$

Exploit the flexibility in the vector potential:

set $\underbrace{\phi = -\dfrac{\vec{\nabla} \cdot \vec{A}}{j\omega\epsilon\mu}}_{0}$

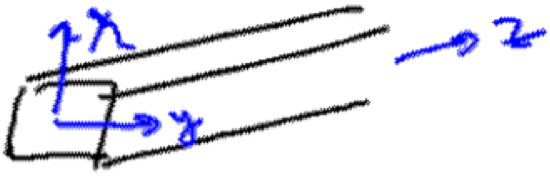
Gauge condition:

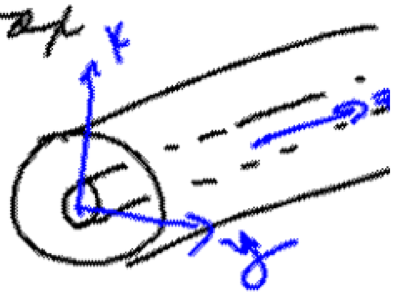
$$\nabla^2 \psi + \omega^2 \epsilon\mu \psi = 0$$

Helmholtz Equation

$\nabla^2 \vec{A} + \omega^2 \epsilon\mu \vec{A} = 0$ 3 uncoupled eqs : A_x, A_y, A_z

Waveguide or coax





$\nabla^2 \vec{A} + \omega^2 \epsilon \mu \vec{A} = 0$ 3 uncoupled eqs: A_x, A_y, A_z

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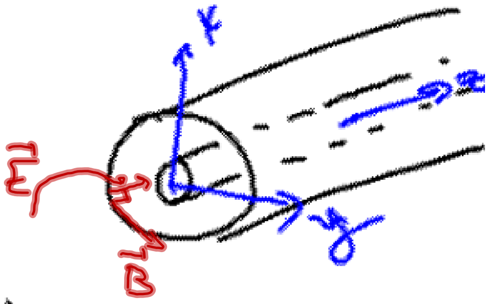
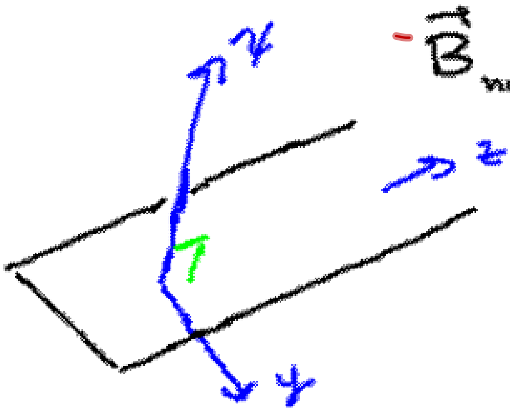


Boundary conditions: $\vec{E}_{tan}\Big|_{wall} = 0$

$\vec{B}_{normal}\Big|_{wall} = 0$

Bottom Wall:



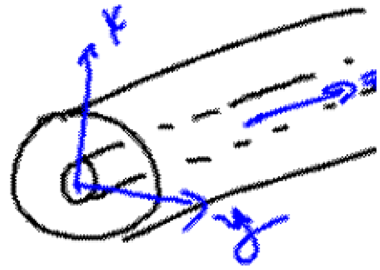
at wall,

$E_y = E_z = 0$

$B_x = 0$

$\nabla^2 \vec{A} + \omega^2 \epsilon\mu \vec{A} = 0$ 3 uncoupled eqs: A_x, A_y, A_z

Waveguide or coax



Boundary conditions: $\vec{E}_{tan}\Big|_{wall} = 0$

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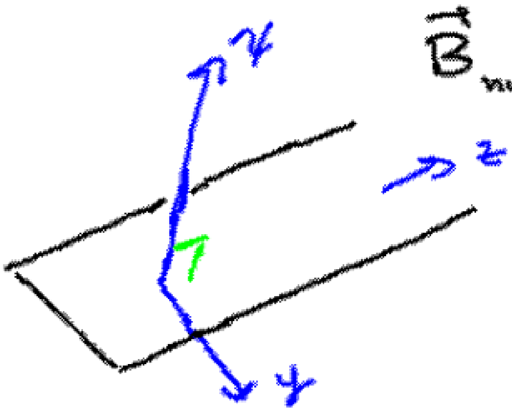
Bottom Wall:



at wall,

$E_y = E_z = 0$

$B_x = 0$

Helmholtz Trick: take $\vec{A} = \psi \vec{k}$

~~Clee~~ Claim: this is a Transverse Magnetic wave;

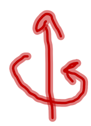
$$\vec{B} = \vec{\nabla} \times \vec{A} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial_x & \partial_y & \partial_z \\ 0 & 0 & \psi \end{vmatrix} = \vec{i}\,\partial_y \psi + \vec{j}\,(-\partial_x \psi) + \vec{k}\,(0)$$

We have already imposed 1 condition on vector potential, which is OK. This new condition is restrictive - we will not get the most general solution.

$$\nabla^2 \psi + \omega^2 \epsilon\mu \psi = 0$$ Helmholtz Equation

Try separating $\psi = \psi_t(x,y) Z(z)$

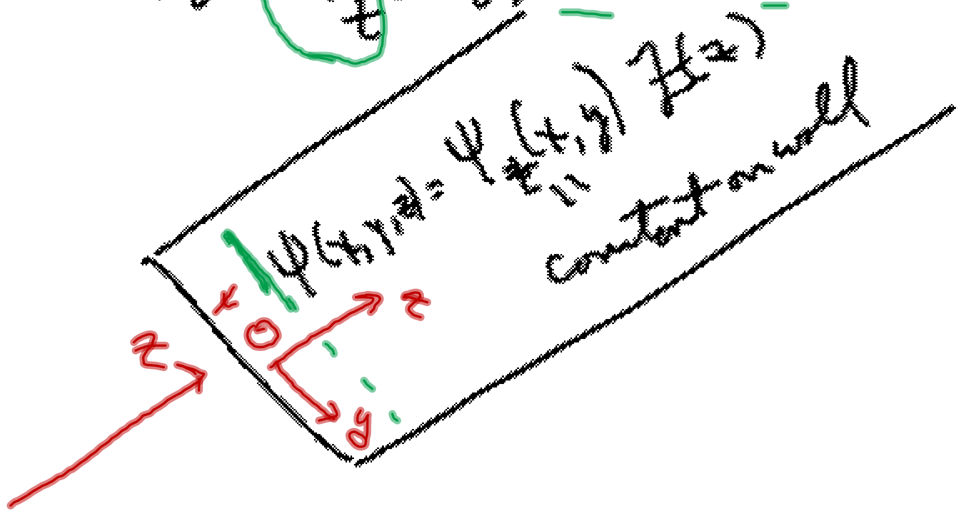
curl



$$BC \Longrightarrow B_n = 0 = B_x = \frac{\partial}{\partial y}\psi \quad \text{on the wall},$$

$$= \frac{\partial}{\partial y}\psi_t(x,y)\,Z(z)$$

$$\text{so} \quad \psi_t = \text{constant on the wall.}$$



$$\cancel{\nabla^2_{2D}\,\psi_t(x,y) + \omega^2 \ldots \psi_t(x,y) = 0}, \quad \psi_t = \text{constant on each wall}$$

$$\nabla^2_{2D}\,\psi_t = \lambda_1\,\psi_t \,,$$

2 dimensional Helmholtz Equation

$$\nabla^2 A_z + \omega^2 \epsilon \mu A_z = 0$$

$$A_z = \psi(x,y,z) = Z(z)\,\psi_t(x,y)$$

$$\frac{\partial^2}{\partial z^2} A_z = Z''\psi_t$$

$$\underbrace{\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)}_{\nabla^2_{2\text{-}D}} A_z = Z\,\nabla^2_{2\text{-}D}\psi_t$$

$$Z(z)\,\nabla^2_{2\text{-}D}\psi_t(x,y) + Z''\psi_t + \omega^2\epsilon\mu\,\psi_t Z = 0$$

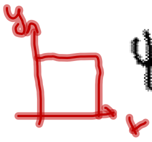
$\div Z\psi_t$

$$\frac{\nabla^2_{2\text{-}D}\psi_t}{\psi_t(x,y)} + \frac{Z''}{Z} + \omega^2\epsilon\mu = 0$$

$$\boxed{\frac{\partial^2}{\partial x^2}\psi_t + \frac{\partial^2}{\partial y^2}\psi_t} \leftarrow \frac{\nabla^2_{2\text{-}D}\psi_t}{\psi_t} = -\frac{Z'}{Z} - \omega^2\epsilon\mu$$

$$(x,y) = (\;z\;) = \text{constant} = \lambda,$$

So far: $\psi(x,y,z) = \psi_t(x,y)\, Z(z)$

2 dimensional Helmholtz Equation $\nabla^2 \psi_t = \lambda_1 \psi_t$  $\psi_t = \text{const.}$ $\left(\vec{B}_{\text{normal}} = 0\right)$

Now enforce $E_x = E_y = 0$ on walls.

$$\vec{E} = -j\omega \vec{A} - \vec{\nabla}\Big(\underbrace{-\frac{1}{j\omega\epsilon\mu}\vec{\nabla}\cdot\vec{A}}_{\phi}\Big)$$

$\vec{A} = \psi \vec{k}$

$\frac{\partial}{\partial z}\psi = \frac{\partial}{\partial z}\psi_t(x,y)\, Z(z)$

use form $Z(z) = e^{\pm\sqrt{\lambda_2} z}$

instead of $\cos\sqrt{\lambda_2}z$, $\sin$

$\psi_t Z$

$$= -j\omega \psi \vec{k} + \frac{1}{j\omega\epsilon\mu}\vec{\nabla}\underbrace{\left(\pm\sqrt{\lambda_2}\psi_t Z\right)}$$

$=$

$\pm\sqrt{\lambda_2}\, Z\, \vec{\nabla}_{2D}\psi_t + \left(\pm\sqrt{\lambda_2}\right)^2 \psi_t Z \vec{k}$

$$\vec{E} = \vec{k}\left[\psi_t e^{\pm\sqrt{\lambda_2}z}\left(j\omega + \frac{1}{j\omega\epsilon\mu}\left[\pm\sqrt{\lambda_2}\right]^2\right)\right]$$

$$\pm\frac{\sqrt{\lambda_2}}{j\omega\epsilon\mu} e^{\pm\sqrt{\lambda_2}\,z}\, \underbrace{\vec{\nabla}_{2D}\psi_t(x,y)}$$

Need to enforce $\vec{E}_{\tan} = 0$

Tang. comp of this is zero on wall; since ψ_t = const on wall, $\vec{\nabla}_{2D}\psi_t$ is $\perp$ to the wall.

* $\psi_t(x,y)\, e^{\pm\sqrt{\lambda_2} z} \left(j\omega + \frac{1}{j\omega\epsilon\mu}\left[\pm\sqrt{\lambda_2}\right]^2 \right)$

need to make * = 0 on wall, since E_z <u>IS</u> a
a tangential component.

Easy way $\psi_t = 0$ [redacted] on all walls

(What's left of the 3 dimensional Helmholtz Equati

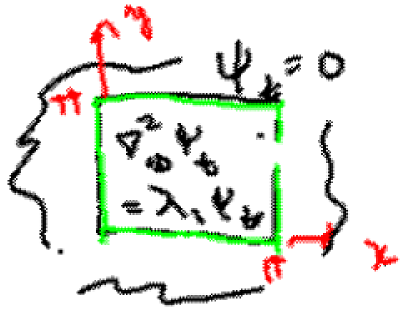
$$\lambda_1 + \lambda_2 + \omega^2\epsilon\mu = ($$

2 dimensional
Helmholtz Equation

$$\nabla^2_{2D}\psi_t = \lambda_1 \psi_t$$

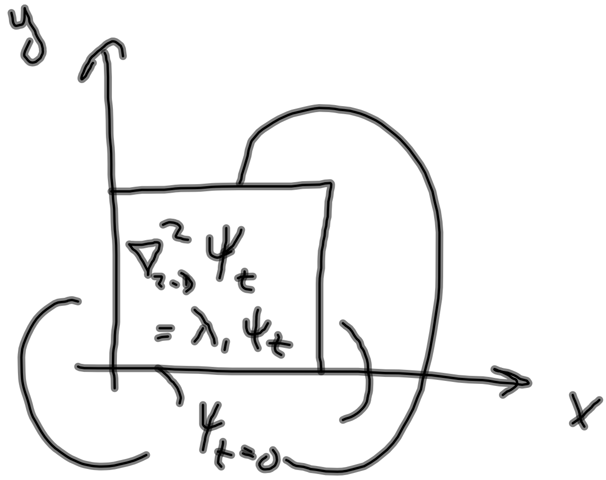
$\psi_t = 0$ on wall

Single conductor



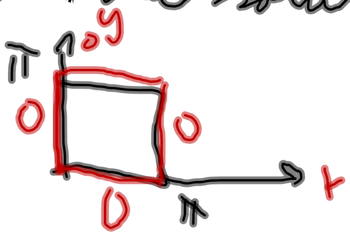
$\psi_t = \sin x \sin y$ <u>works</u>!

$$\nabla^2_{2D}\psi_t = -\sin x \sin y - \sin x \sin y$$
$$= -2\psi_t . \quad \lambda_1 = -2$$



~~Is the~~ Can this have non-trivial solutions?

yes: $\psi_t = (\sin x \ \sin y)$



✓ $\nabla^2_{2D}\psi_t = -\psi_t - \psi_t = -2\psi_t$

$$* \; E_z = \psi_t(x,y)\, e^{\pm\sqrt{\lambda_2} z}\left(j\omega + \frac{1}{j\omega\epsilon\mu}\left[\pm\sqrt{\lambda_2}\right]^2\right)$$ must equal zero on wall

Hard way $\quad -j\omega + \frac{1}{j\omega\epsilon\mu}\left[\pm\sqrt{\lambda_2}\right]^2 = 0$

$$\Rightarrow \lambda_2 = -\omega^2\epsilon\mu.$$

Note: E_z will be zero everywhere: this field is also T_v E_{lectric}.

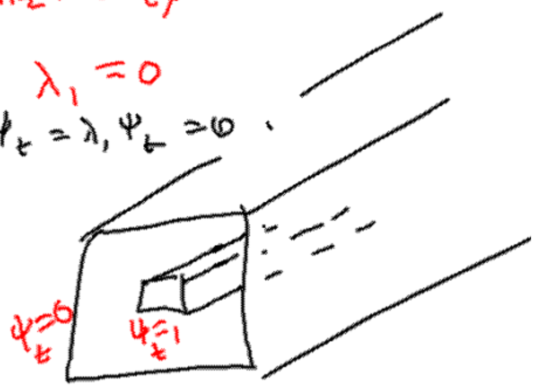
~~Ans~~ Don't care what constant ψ_t is on the wall.

$$\lambda_1 + \lambda_2 + \omega^2\epsilon\mu = 0$$

$$\Rightarrow \lambda_1 = 0$$

$$\nabla^2_{xy}\psi_t = \lambda_1\psi_t = 0.$$

GREAT!



Solve $\nabla^2_{xy}\psi_t = 0$ with these BC's,

$\Rightarrow \psi_t(x,y)$.

Attach $Z(z) = e^{\pm\sqrt{\lambda_2}\, z}$

$$= e^{\pm j\omega\sqrt{\epsilon\mu}\, z}$$

$$= \begin{matrix}\cos\\ \sin\end{matrix}\left(\omega\sqrt{\epsilon\mu}\, z\right)$$

Works for any ω down to 0 (D.C)

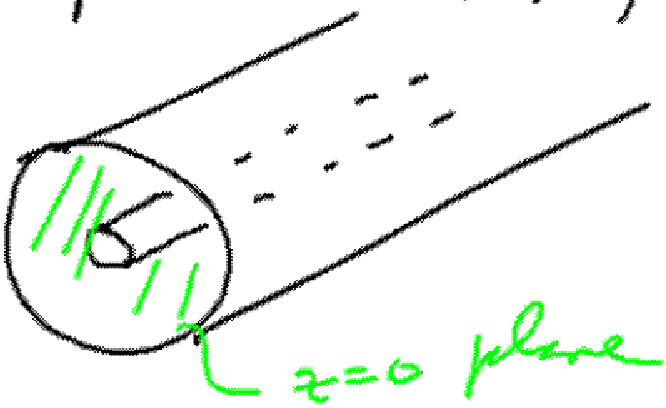
$$\psi = \psi_t(x,y)\cos\left[\omega\sqrt{\epsilon\mu}\, z - \omega t\right]$$

Non-dispersive traveling wave, going down the waveguide at speed

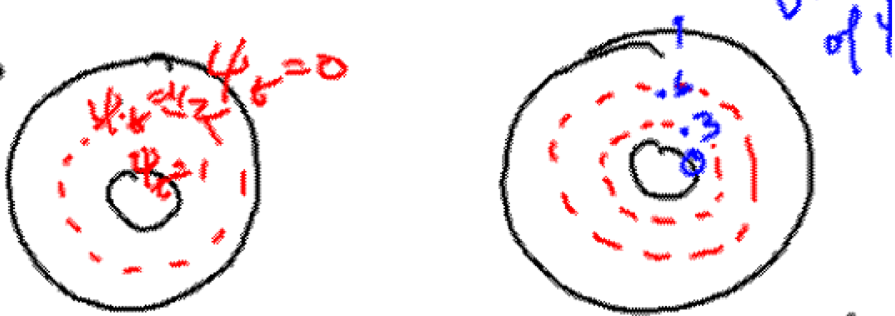
wave front $\quad \omega\sqrt{\epsilon\mu}\, z - \omega t = \text{constant} = 0$

$$z/t = \frac{1}{\sqrt{\epsilon\mu}} = \text{speed}$$

$E_z=0$ $B_z=0$ for these solns; They are TEM



In z=0 plane,

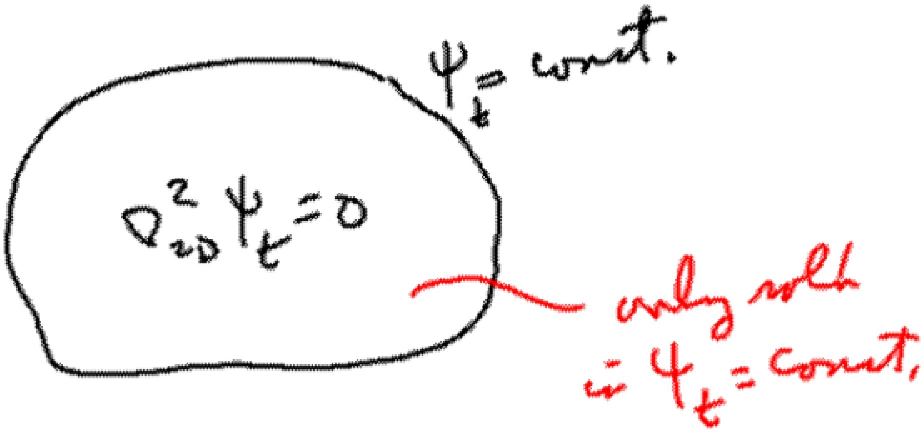


To get the field at other z, multiply this ~~pl~~ pattern by $\cos[\omega(\sqrt{\epsilon\mu}\,z - t)]$ or sin, ...

If $\omega=0$, $\cos(0)=1$; static, DC, z-invariant solution.

~~With this strategy, need to solve~~

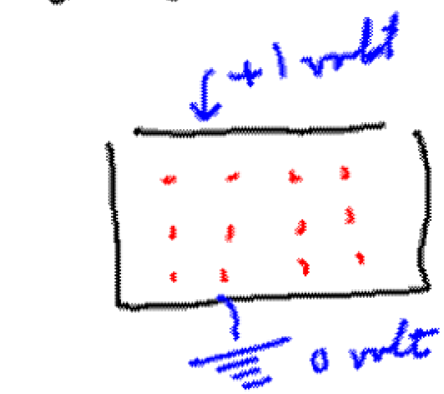
With a single conductor in this model you must solve



So this (TEM) sol'n is available only for a multiple conductor.

True waveguides have no TEM wave.

Case 1 with:



Solve the DC potential:

$\nabla^2 \phi = 0$, $\phi = \begin{cases} 1 & \text{on top} \\ 0 & \text{on bottom} \end{cases}$

$\phi = \phi(x,y)$

Use finite difference, Laplace molecule, relaxation.

attach $e^{i(\omega t - kz)}$ $k = \omega\sqrt{\epsilon\mu}$ to $\phi(x,y)$.

Core 2 math.

$\Psi_t = 0$

$\nabla^2_{2D}\Psi_t = \lambda_1 \Psi_1$

Solve

$$\boxed{\nabla^2_{2\text{-}D}\psi_t(x,y) = \lambda_1 \psi_t(x,y)}$$

Slap on factor $e^{j(\omega t - Kz)}$

$$K = \sqrt{\omega^2 \epsilon\mu + \lambda_1}$$

This problem only has nontrivial solution for discrete set of eigenvalues λ_1. Typically $0 > \lambda_1 > \lambda_2 \ldots$

The Cutoff Frequency: Suppose $\lambda_1 = -10$.

$$K = \sqrt{\omega^2 \epsilon\mu - 10}$$

If ω is too low ($\omega^2\epsilon\mu - 10 < 0$, K is imaginary),

e^{-jKz} is $e^{-j^2 \# z}$ = exponential growth or decay.

Lowest cutoff freq. is $\omega^2 \epsilon\mu + \lambda_1 > 0$

$$\cancel{\omega = \sqrt{\frac{\lambda_1}{\epsilon_1}}}$$

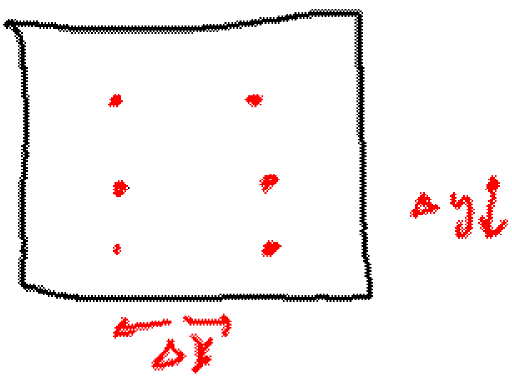
$$\omega = \sqrt{\frac{-\lambda_1}{\epsilon\mu}} \quad (\lambda_1 < 0)$$

How do you solve $\psi_t = 0$ numerically

$$\boxed{\nabla^2 \psi_t = \lambda \psi_t}$$

You don't know λ at the outset. Only privileged values give you a nontrivial solution.

1. Write finite difference equations for $\nabla^2 \psi_t = \lambda \psi_t$
2. Rewrite as a matrix system. It will look like $\begin{bmatrix} A \end{bmatrix} \begin{bmatrix} \psi \\ \vdots \end{bmatrix} = \lambda \begin{bmatrix} \psi \\ \vdots \end{bmatrix}$.
3. Ask MATLAB for values of ψ & λ.



$$\cancel{\nabla^2 \psi = \psi_{\uparrow} + \psi_{\rightarrow} + \psi_{\leftarrow} + \psi_{\downarrow} - 4\psi_0}$$

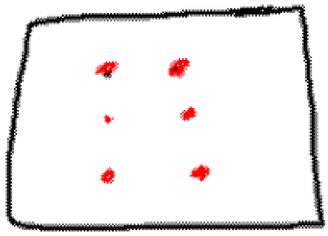
2D

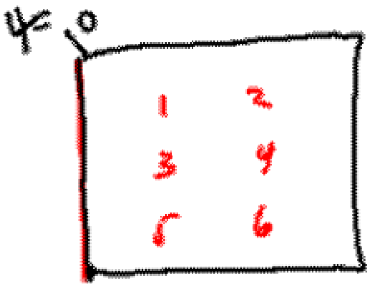
From Sadiku's table

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{\psi_{\rightarrow} + \psi_{\leftarrow} - 2\psi_0}{\Delta x^2} \qquad \frac{\partial^2 \psi}{\partial y^2} = \text{''}$$

$$\nabla^2 \psi \approx \frac{\psi_{\rightarrow} + \psi_{\leftarrow} - 4\psi_0 + \psi_{\uparrow} + \psi_{\downarrow}}{\Delta x^2} + O(\Delta x^2)$$

$\Delta x = \Delta y$.





$$\nabla^2 \psi = \lambda \psi$$

$$\nabla^2 \psi \approx \frac{\psi_{\uparrow} + \psi_{\downarrow} + \psi_{\leftarrow} + \psi_{\rightarrow} \cdots}{\Delta x^2}$$

at node #1 $\dfrac{0 + \psi_3 + 0 + \psi_2 - 4\psi_1}{\Delta x^2} = \lambda \psi_1$

at node #2: $\dfrac{0 + \psi_4 + 0 + \psi_1 - 4\psi_2}{\Delta x^2} = \lambda \psi_2$

#3: $\dfrac{\psi_1 + \psi_5 + \psi_4 + 0 - 4\psi_3}{\Delta x^2} = \lambda \psi_3$

#4: $\dfrac{\psi_2 + \psi_6 + 0 + \psi_3 - 4\psi_4}{\Delta x^2} = \lambda \psi_4$

#5 $\dfrac{\psi_3 + 0 + \psi_6 + 0 - 4\psi_5}{\Delta x^2} = \lambda \psi_5$

#6 $\dfrac{\psi_4 + 0 + 0 + \psi_5 - 4\psi_6}{\Delta x^2} = \lambda \psi_6$

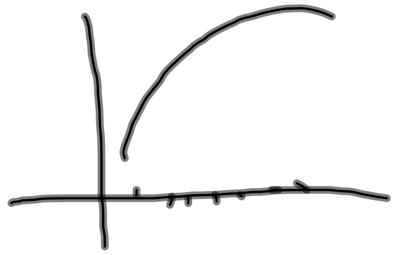
$$\underbrace{\frac{1}{\Delta x^2}\begin{bmatrix} -4 & 1 & 1 & 0 & 0 & 0 \\ 1 & -4 & 0 & 1 & 0 & 0 \\ 1 & 0 & -4 & 1 & 1 & 0 \\ 0 & 1 & 1 & -4 & 0 & 1 \\ 0 & 0 & 1 & 0 & -4 & 1 \\ 0 & 0 & 0 & 1 & 1 & -4 \end{bmatrix}}_{[A]} \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \\ \psi_5 \\ \psi_6 \end{bmatrix} = \lambda \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \\ \psi_5 \\ \psi_6 \end{bmatrix}$$

$$[A][\psi] = \lambda [\psi]$$

New approach to num EM.

We have to solve (approximately) D.E.'s.

$$\frac{d^2y}{dx^2} = 0$$

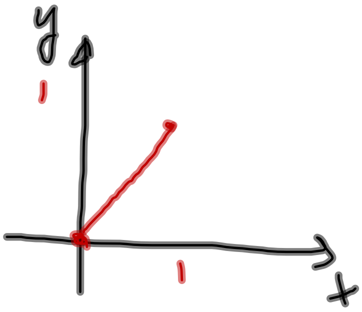


Try to find a near-solution like

$$\Rightarrow \quad y(x) \simeq c_1 e^{x} + c_2 e^{-x} + c_3 1 + c_4 e^{2x}$$

What values of $c_1 - c_4$ gives best approximation to a solution?

$$\frac{d^2y}{dx^2} = 0, \quad y(0) = 0, \quad y(1) = 1$$

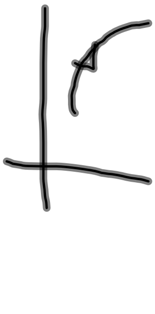


① Enforce BCs; $c_1 \cdots c_4 = 0$ $\quad c_1 \sim c_4 = 1$

Still have 2 c's to fiddle with.

$$y = c_1 e^{x} + c_2 e^{-x} + c_3 + c_4 e^{2x}$$

Fact: $\frac{d^2y}{dx^2} = 0 \iff y(x)$ is the shortest curve between (0,0) & (1,1).



$$\text{distance} = \int_{\text{start}}^{\text{finish}} \sqrt{dx^2 + dy^2} = \int \sqrt{1 + y'^2}\, dx = \cdots c_1 - c_3 \cdots$$

$$\frac{\partial}{\partial c_1} = \frac{\partial}{\partial c_2} = 0$$

Abbreviate $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ by $\nabla^2_{transverse}$

$$\nabla^2 A_z + \omega^2 \varepsilon\mu A_z = 0 = [\nabla^2_{tran}\, \psi_{tran}]\, Z + \psi_{tran}\, Z'' + \omega^2\varepsilon\mu\, \psi_{tran}\, Z$$

$$= \{\psi_{tran}\, Z\}\, \{\, [\nabla^2_{tran}\, \psi_{tran}] / \psi_{tran} + Z''/Z + \omega^2\varepsilon\mu\}$$

$0 =$ (---) {f(x,y) + g(z) + constant} only possible if f and g are constant

So $\boldsymbol{\nabla^2_{tran}\, \psi_{tran} = \lambda_1 \psi_{tran}}$ and $\boldsymbol{Z'' = \lambda_2 Z}$ and $\boldsymbol{\lambda_1 + \lambda_2 + \omega^2\varepsilon\mu = 0}$.

First characterize Z: solutions of $Z'' = \lambda_2 Z$ are $e^{\pm\sqrt{\lambda_2}z}$

Next characterize $\psi_{tran}(x,y)$. $\nabla^2_{tran}\, \psi_{tran} = \lambda_1 \psi_{tran}$

$\psi_{tran}(x,y)$ is constant along each section of the wall (because $B_{normal} = 0$)

What about E [illegible] = 0 ? From page 1

$$\mathbf{E} = \nabla_{tran}\{\psi_{tran}\}Z'/j\omega\varepsilon\mu \quad + \quad \psi_{tran}\, Z\{\cancel{-j\omega + \lambda_2/j\omega\varepsilon\mu}\}\,\mathbf{k}\,. \qquad [\,\mathbf{E}_{tan} = 0\,]$$

The tangential gradient of ψ_{tran} is already zero on the wall, since ψ_{tran} is constant there

And since E_z is also a tangential component at the wall, we have two choices:

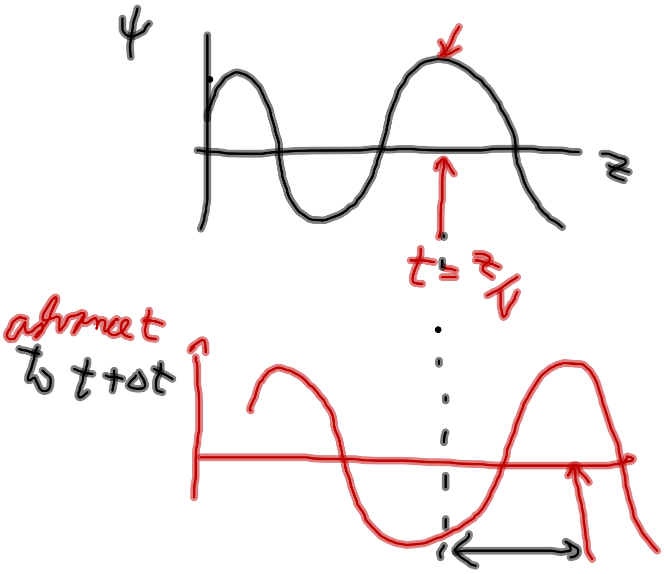
$$\lambda_2 = -\omega^2\varepsilon\mu \qquad \text{or} \qquad \psi_{tran}(x,y) = 0 \text{ on all wall sections.}$$

First choice.

$$\sqrt{\lambda_2} = \pm j\omega\sqrt{\varepsilon\mu} \;\; and \;\; \lambda_1 = -\lambda_2 - \omega^2\varepsilon\mu = 0$$

Dispersion.

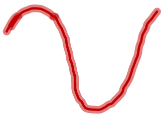
$$\psi = \cos \omega\left(t - \frac{z}{v}\right)$$



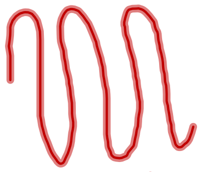
$$t + \Delta t - \frac{z + \Delta z}{v} = 0$$

$$\Delta z = \Delta t \cdot v$$

Suppose $$\cos \omega_1\left(t - \frac{z}{v_1}\right) + \cos \omega_2\left(t - \frac{z}{v_1}\right)$$



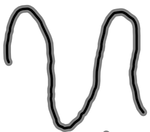
$\rightarrow$ at speed v_1



$\rightarrow$ at speed v_1

But

$$\cos \omega_1\left(t - \frac{z}{v_1}\right) + \cos \omega_2\left(t - \frac{z}{v_2}\right)$$



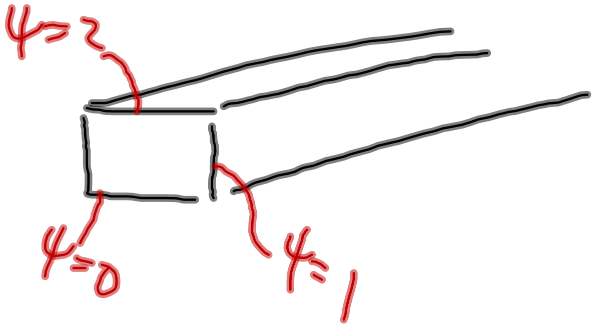
$\rightarrow$ at speed v_1

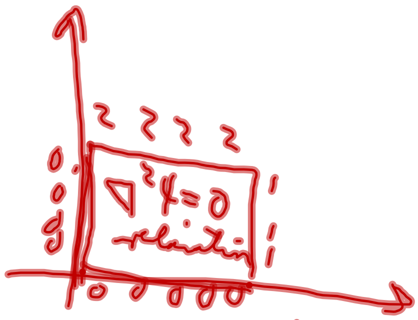
$\rightarrow$ at speed v_2

DISTORT

Computational Technique for TEM.

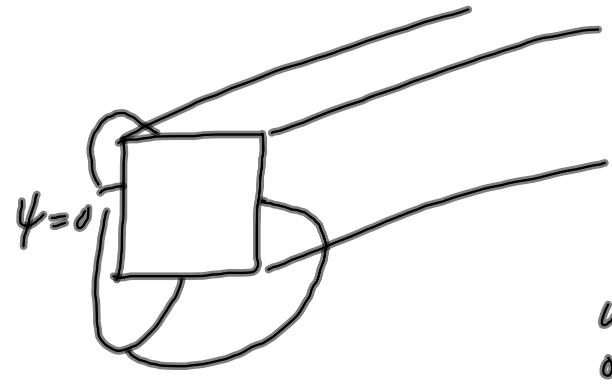
multiple conductors, $\psi_{\text{TEM}}(x,y)$ constant on each conductor

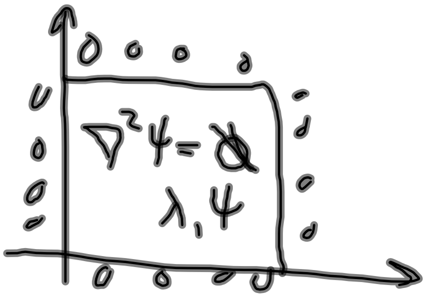




$\psi(x,y)\, e^{j\omega(t - z\sqrt{\epsilon\mu})}$

To solve TM





λ_1 is unknown.

Technique: go to the matrix formulation

$$\underbrace{\begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \ddots & \cdot \\ & & \end{bmatrix}\begin{bmatrix}\psi\\ \vdots \\ \end{bmatrix} = \lambda_1 \begin{bmatrix}\psi\\ \vdots \\ \psi\end{bmatrix}}$$

$A\,v = \lambda v$

eigenvalue prob.

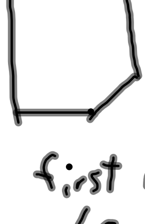
ask MATLAB eig(A)

$$\Rightarrow \lambda\begin{bmatrix}\psi\\ \psi\\ \psi\end{bmatrix}, \lambda\begin{bmatrix}\psi\\ \vdots\end{bmatrix}, \ldots$$

$$0 > \lambda > \lambda > \lambda > \ldots$$

In theory, ∞ # of λ's.

On computer, $\left[\quad\right]_{N \times N}$ $N\ \lambda$'s.

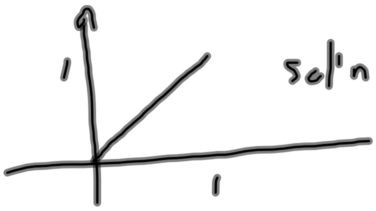


first cutoff freq.
(first eigenvalue)

~~What's to~~

Variational Formulation.

DE $\dfrac{d^2y}{dx^2}=0$ BC $y(0)=0$ $y(1)=1$



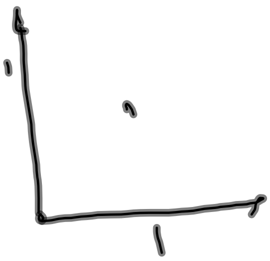
sol'n $y(x)=x$

Try alternative approach.

The DE $\Longrightarrow$ sol'n is a straight line,
shortest distance between 2 points.

Change the prob.

Find the shortest path between
$(0,0)$ & $(1,1)$



Rayleigh-Ritz procedure for finding shortest paths:

try a shape function

$$y = a + bx + cx^2 + d\sin x$$

a, b, c, d to be determined:

① BC: $y(0)=0$, $y(1)=1$ eliminate c and d

② distance $= \int \sqrt{dx^2+dy^2} = \int_{x=0}^{1} \sqrt{1+y'^2}\,dx$

work out integral with a, b, c, d.

$$\int_0^1 \sqrt{1+y'^2}\,dx = I(a,b,c,d) \quad \text{Known formula}$$

Choose a, b, c, d to minimize the length.

$$\frac{\partial I}{\partial a}=0 \quad \frac{\partial I}{\partial b}=0 \quad \cancel{\frac{\partial I}{\partial c}}=0 \quad \cancel{\frac{\partial I}{\partial d}}=0$$

So ~~the~~ Rayleigh-Ritz gives you a fcn. which does the best to minimizing the integral (length), of all functions of the form you chose.

Issues:

1. The form you use has to be flexible enough to approximate the actual solution.
2. You have to be able to do the integrals.
3. The form has to be reasonably simple.

<u>Example</u> $y'' = 0$, $y(0) = 0$, $y(1) = 1$

choose $y(x) = a\_\_ + b\_ + c\_$ (next time?)

"basis functions"

Evaluate $\int_0^1 \sqrt{1 + y'^2}\, dx$

Issue:

DE $y''=0$ is related to min $\int \sqrt{1+y'^2}\,dx$

What about

$\nabla^2 V = 0$ " " " ?

$\nabla^2 \psi + \omega^2 \epsilon\mu \psi = 0$ " " " ?

Yes.

Techniques to be covered:

all involve choosing a form for your approximation

$$\psi = a\,\_\_\_ + b\,\_\_\_ + c\,\_\_\_ + \cdots$$

basis functions

① Variational Method: replace P E by min $\int \cdots\cdots$.

Choose a, b, c, to minimize the integral.

"Rayleigh-Ritz"

$$\psi = a\,(\text{basis } 1) + b\,(\text{basis } 2) + \underbrace{c\,(\text{---})\,\ldots}_{n \text{ coefficients}}$$

D.E. LHS = RHS

2. Choose a, b, c .. so that LHS = RHS n times

"Collocation"

3. LHS = RHS $\Longrightarrow$ LHS − RHS = 0

$$\text{instead, } \underbrace{\text{minimize}}_{\text{choose } a_1 - a_n} \int [\text{LHS} - \text{RHS}]^2$$

4. LHS = RHS

Plug in $y = a_1 \text{---} + a_2 \text{---} \ldots$

LHS $\approx$ RHS

multiply both sides by, say, $\sin x$ and integrate

$$\int \text{LHS} \cdot \sin x = \int \text{RHS} \cdot \sin x$$ "test functions"

and $$\int \text{LHS} \cdot \sin 2x = \int \text{RHS} \cdot \sin 2x$$

and $$\int \text{LHS} \cdot x = \int \text{RHS} \cdot x$$

Boundary / Interface Conditions

$h \ll r$, $h \ll l$

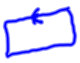
all dimensions $\to 0$

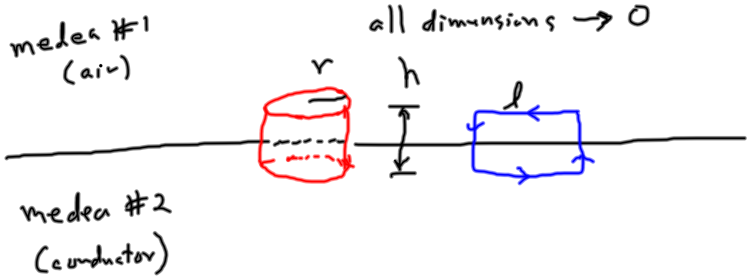
For $\vec{\nabla} \cdot \vec{F} = \cdots$,

use Gaussian pillbox

For $\vec{\nabla} \times \vec{F} = \cdots$,

use Amperian loop





Volume charge density

$$\nabla \cdot \vec{D} = \rho_v \Longleftrightarrow \vec{\nabla} \cdot \vec{E} = \rho_v / \epsilon_0$$

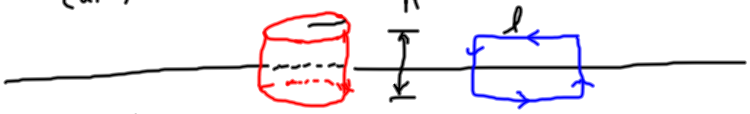
$$\nabla \cdot \vec{B} = 0$$

$h << r$, $h << l$

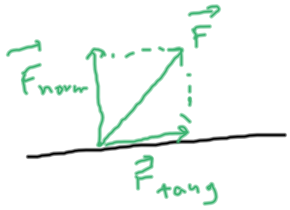
all dimensions $\rightarrow 0$

For $\vec{\nabla} \cdot \vec{F} = \cdots$, use Gaussian pillbox

medea #1 (air)



medea #2 (conductor)



$\vec{\nabla} \cdot \vec{F}$ = outflux per unit volume

$\therefore$ outflux $= \pi r^2 \cdot h \cdot (\nabla \cdot \vec{F})$

Since $h << r$, outflux $\approx F^{(1)}_{normal} \pi r^2 - F^{(2)}_{norm} \pi r^2$

So $F^{(1)}_{norm} - F^{(2)}_{norm} = \dfrac{\pi r^2 \cdot h \, (\nabla \cdot \vec{F})}{\pi r^2}$

$\therefore$ $B^{(1)}_{norm} - B^{(2)}_{norm} = 0$ $\quad$ $\vec{B}_{normal}$ is continuous across an interface

$$E^{(1)}_{norm} - E^{(2)}_{norm} = \frac{\pi r^2 h \rho_v / \epsilon_0}{\pi r^2} = \frac{(\text{charge enclosed}) / \epsilon_0}{(\text{area})}$$

$$= \rho_s / \epsilon_0$$

$\uparrow$ surface charge density

$\therefore$ $B^{(1)}_{norm} - B^{(2)}_{norm} = 0$ $\quad$ $\vec{B}_{normal}$ is continuous across an interface

$$E^{(1)}_{norm} - E^{(2)}_{norm} = \frac{\pi r^2 h \rho_v / \epsilon_0}{\pi r^2} = \frac{(\text{charge enclosed}) / \epsilon_0}{(\text{area})}$$

$$= \rho_s / \epsilon_0$$

$\uparrow$ surface charge density

① If E_n has a jump across an interface, there is a surface charge density $= \Delta E_n \cdot \epsilon_0$

② If the fields are zero in medea #2,

$B^{(1)}_n = 0$ $\qquad$ $E^{(1)}_n = \rho_s / \epsilon_0$

(This occurs if medea #2 is a perfect conductor and no DC; even at DC if temperature $= 0^\circ K$.)

$$\vec{\nabla}\times\vec{E} = -j\omega\vec{B}$$

volume current density

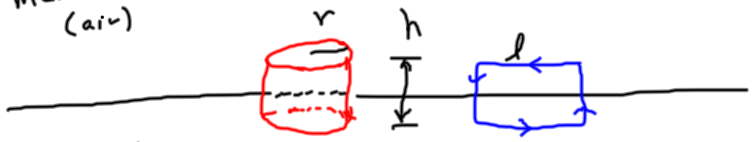
$$\vec{\nabla}\times\vec{H} = j\omega\vec{D} + \vec{J}_v \iff \vec{\nabla}\times\vec{B} = j\omega\epsilon\mu\vec{E} + \mu\vec{J}_v$$

$h \ll r$, $h \ll \ell$

all dimensions $\rightarrow 0$

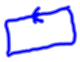
For $\vec{\nabla}\cdot\vec{F} = \cdots$,

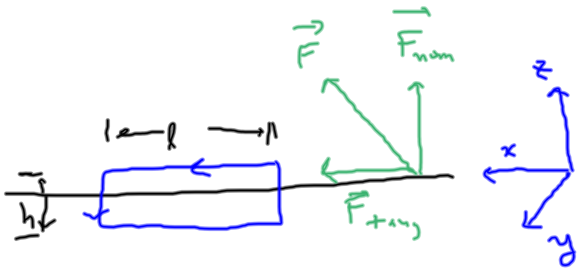
use Gaussian pillbox



For $\vec{\nabla}\times\vec{F} = \cdots$,

use Amperian loop





$$\vec{\nabla}\times\vec{F} = \oint \vec{F}\cdot d\vec{\ell} \Big/ \text{area}$$

For $h \ll \ell$, $\oint \vec{F}\cdot d\vec{\ell} = F_x^{(1)}\ell - F_x^{(2)}\ell = (\text{area})\left(\vec{\nabla}\times\vec{F}\right)_y$

$\vec{\nabla}\times\vec{E} = -j\omega\vec{B}$

$$E_x^{(1)} - E_x^{(2)} = \frac{\ell h\,(-j\omega B_y)}{\ell} = h\,(-j\omega B_y) \rightarrow 0$$

$\vec{E}_{tang}$ is continuous across an interface.

$\vec{\nabla}\times\vec{B} = j\omega\epsilon\mu\vec{E} + \mu\vec{J}_v$

$$B_x^{(1)} - B_x^{(2)} = \underbrace{\frac{\ell h(j\omega\epsilon\mu E_y)}{\ell}}_{\searrow\, 0} + \underbrace{\mu\frac{\overbrace{(\ell h)J_{v,y}}^{\text{total current}}}{\ell}}_{\searrow\, \mu J_{s,y}}$$

surface current density

$$\vec{B}^{(1)}_{tang} - \vec{B}^{(2)}_{tang} = \mu\vec{J}_s$$

If there is a discontinuity in $\vec{B}_{tang}$, there is a surface current density $\vec{J}_s = \Delta\vec{B}_{tang}/\mu$

If all fields in medea #2 are zero,

$$\vec{E}^{(1)}_{tang} = 0 \qquad \vec{B}^{(1)}_{tang} = \mu\vec{J}_s.$$

In-class midterm exam.

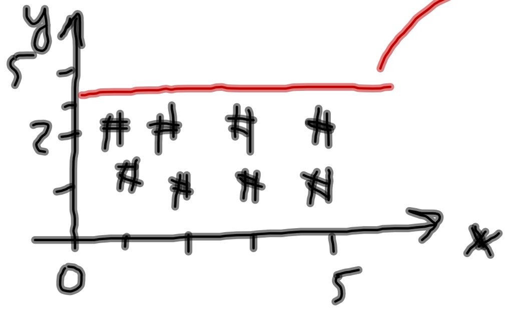
March 26, 2009.

Typical.

What is $\frac{\partial \psi}{\partial y}$ at ?, 2 ? at a boundary?

" " $\nabla^2 \psi$ — ? Watch out! $\Delta x \neq \Delta y$

Typical.



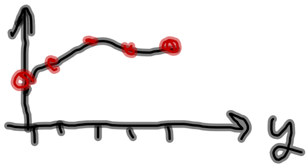
What is $\int_0^5 f(x,3)\,dx$?

(Simpson's Rule)

What is $\int_0^5\int_0^5 f(x,y)\,dx\,dy$?

$$= \int_0^5 \left[\int_0^5 f(x,y)\,dx \right] dy$$

outer integral =



What is $\int_{y=0}^{5} \int_{x=0}^{y} 4(x,y)\, dx\, dy$

Collocation, Variational to PE

Approx.

<u>Collocation.</u> DE $y''=0$ BC's: $y(0)=0$, $y(1)=1$

(soln $y=x$).

Basis functions: $1, \cancel{x}, x^2, x^3, x^4$

Try: $y = c_1 + \cancel{c_2 x} + c_3 x^2 + c_4 x^3 + c_5 x^4$

B.C.'s: $y(0)=0 \Rightarrow c_1 = 0$ $\quad y = c_3 x^2 + c_4 x^3 + c_5 x^4$

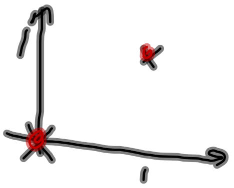
$y(1)=1$ $\quad c_5 = -c_4 - c_3 + 1$

$$y = c_3 x^2 + c_4 x^3 + (1 - c_3 - c_4)x^4$$

DE: $y'' = 0$

$$2c_3 + 3\cdot 2 c_4 x + 12(1 - c_3 - c_4)x^2 \overset{\text{D.E.}}{\equiv} 0$$

Can't make it work at every x.
So (c_3, c_4) make it work at 2 locations.



Try $x = 1/3$, $x = 2/3$

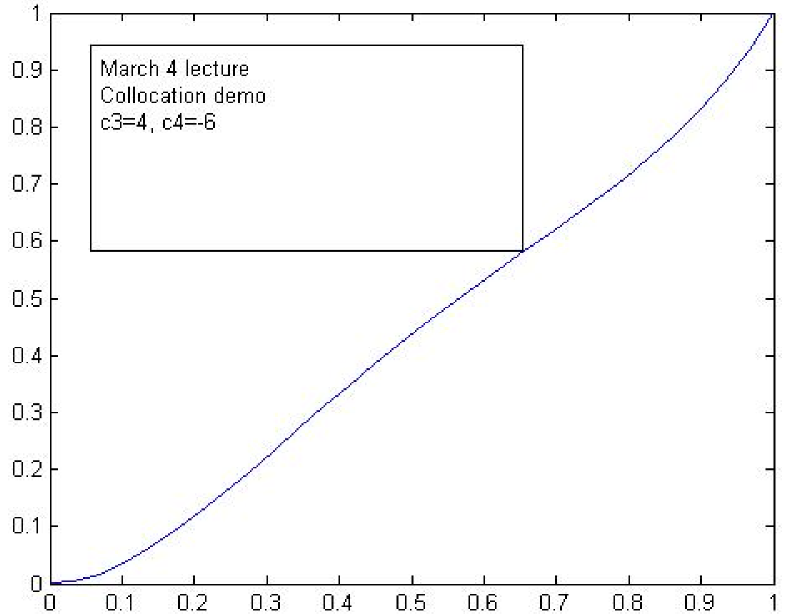
y = deg-4 poly.
Expon
no X

$$2c_3 + 6c_4 \cdot \tfrac{1}{3} + 12(1 - c_3 - c_4)\left(\tfrac{1}{3}\right)^2 = 0$$

$$2c_3 + 6c_4 \cdot \tfrac{2}{3} + 12(1 - c_3 - c_4)\left(\tfrac{2}{3}\right)^2 = 0$$

2 eqs., 2 unknowns, linear.

The Answer is in the mail.


March 4 lecture
Collocation demo
c3=4, c4=-6
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

Variational formulation.

DE: $y''=0$ B.C. $y(0)=0$, $y(1)=1$

We saw that this minimizes $\int_0^1 \sqrt{1+y'^2}\,dx$.

I will show you that it minimizes $\int_0^1 y'^2\,dx$

Then I will choose c_3, c_4 so that

$$y = c_3 x^2 + c_4 x^3 + (1 - c_3 - c_4) x^4$$

minimizes $\int_0^1 y'^2\,dx$.

~~1st~~ Theory: find conditions on $y(x)$ such that

$y(0)=0$, $y(1)=1$, $\int_0^1 y'^2\,dx$ is minimal.

Suppose $y_0(x)$ is the function that does minimize $\int_0^1 y'^2\,dx$ & $y_0(0)=0$, $y_0(1)=1$

Perturb $y_0(x)$ by $\eta(x)$: $y(x) \doteq y_0(x) + \eta(x)$

(condition: $\eta(0)=0$
$\eta(1)=0$)

Regardless of $\eta(x)$,

$$\int_0^1 y'^2\,dx \geq \int_0^1 y_0'^2\,dx$$

$$\| $$

$$\int_0^1 [y_0(x)+\eta(x)]'^2\,dx$$

$$\| $$

$$\int_0^1 [y_0'^2 + 2y_0'\eta' + \eta'^2]\,dx \geq \int y_0'^2\,dx$$

η is small; neglect η^2, η'^2.

$$\int_0^1 \left[\cancel{y_0'^2} + 2y_0'\eta' + \cancel{\eta'^2}\right] dx \geq \int \cancel{y_0'^2}\, dx \quad 0$$

η is small; neglect η^2, η'^2.

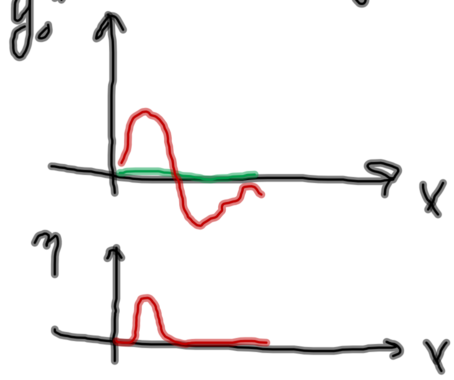
$$\cancel{2}\int_0^1 y_0'\eta'\, dx \geq 0 \quad \text{regardless of } \eta \quad (\eta(0)=\eta(1)=0)$$

$$\int_0^1 u\, v'\, dx = uv\Big|_0^1 - \int v u'\, dx$$

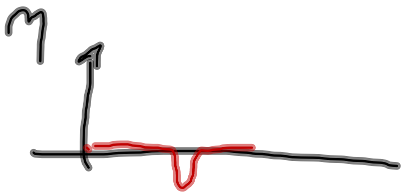
$$\int_0^1 y_0'\eta'\, dx = \underbrace{y_0'\eta\Big|_0^1}_{0} - \int \eta(x)\, y_0''(x)\, dx \geq 0$$

Conclude $-\int \eta(x)\, y_0''(x)\, dx \geq 0$ for any $\eta(x)$ $(\eta(0)=\eta(1))$

It follows that $y_0'' \equiv 0$:



violates $-\int y_0''\eta \geq 0$



violates $-\int y_0''\eta \geq 0$

Conclusion : $y_0''(x) = 0$ D. E.

~~So try~~

So try $y = c_3 x^2 + c_4 x^3 + (1 - c_3 - c_4) x^4$

$\int y'^2 dx$ $\qquad y' = 2c_3 x + 3c_4 x^2 + 4(1 - c_3 - c_4) x^3$

$$y'^2 = 4c_3^2 x^2 + 9c_4^2 x^4 + 16(1 - c_3 - c_4)^2 x^6$$

$$+ 2 \cdot 2c_3 x \cdot 3c_4 x^2 + 2 \cdot 3c_4 x^2 \cdot 4(1 - c_3 - c_4) x^3$$

$$+ 2 \cdot 2c_3 x \cdot 4(1 - c_3 - c_4) x^3$$

$$\int_0^1 y'^2 dx = 4c_3^2 \cdot \tfrac{1}{3} + 9c_4^2 \tfrac{1}{5} + 16(1 - c_3 - c_4)^2 \tfrac{1}{7}$$

$$+ 12 c_3 c_4 \tfrac{1}{4} + 24 c_4 (1 - c_3 - c_4) \tfrac{1}{6}$$

$$+ 16 c_3 (1 - c_3 - c_4) \tfrac{1}{5}$$

$$\int_0^1 y'^2\,dx = 4c_3^2 \cdot \frac{1}{3} + 9c_4^2 \frac{1}{5} + 16\,(1-c_3-c_4)^2 \frac{1}{7}$$

$$+ 12\,c_3 c_4 \frac{1}{4} + 24\,c_4(1-c_3-c_4)\frac{1}{6}$$

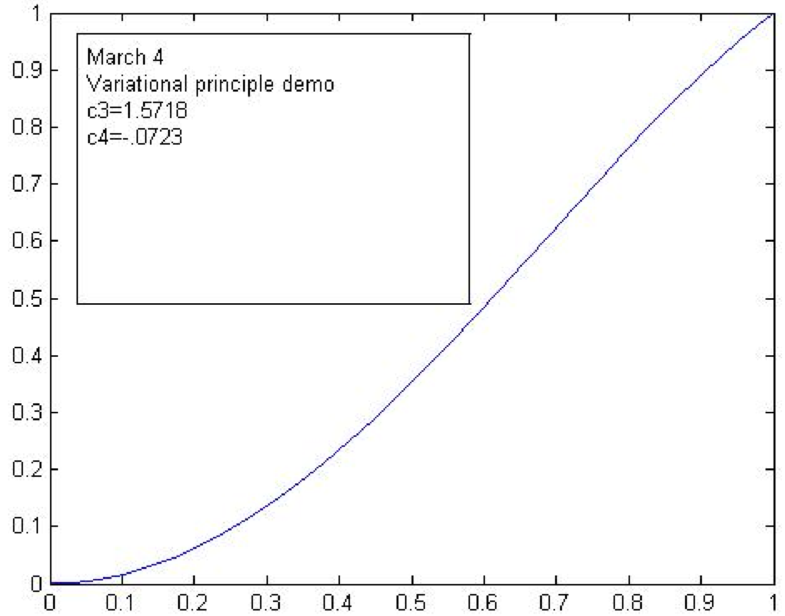
$$+ 16\,c_3(1-c_3-c_4)\frac{1}{5}$$

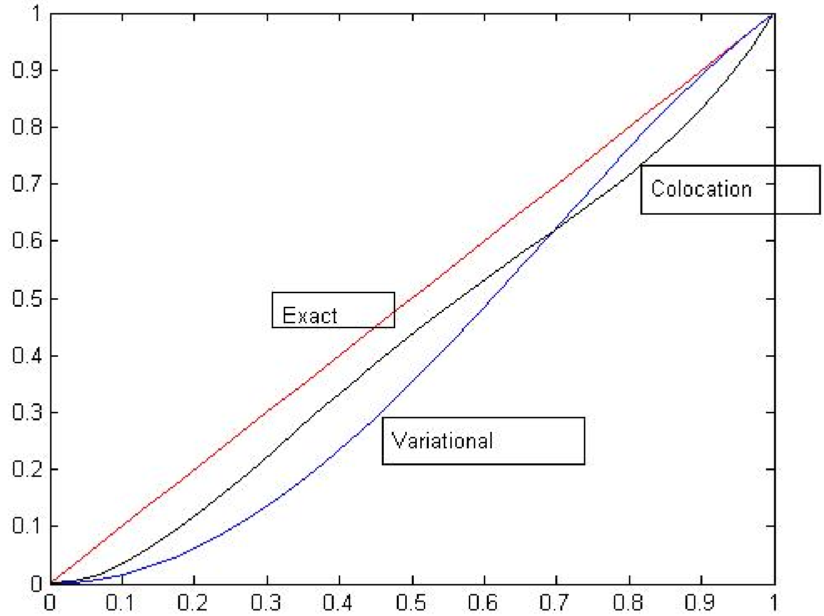
minimize

$$0 = \frac{\partial(\ )}{\partial c_3} = 2\cdot 4\cdot c_3 \cdot \frac{1}{3} + 0 + 2\cdot 16\,(1-c_3-c_4)(-1)\frac{1}{7} + \dots$$

$$0 = \frac{\partial(\ )}{\partial c_4} = \dots \text{ linear}(c_3, c_4)$$

Solve: get (c_3, c_4)


March 4
Variational principle demo
c3=1.5718
c4=-.0723


Exact
Variational
Colocation
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

Generalize: The function which minimized $\int_0^1 y'^2\,dx$, $y(0)=0$, $y(1)=1$,

satisfies DE: $y''=0$

Euler-Lagrange theory $\Rightarrow$

The function which minimizes $\int_a^b \mathcal{L}[y',y,x]\,dx$

satisfies the DE

$$\frac{d}{dx}\frac{\partial \mathcal{L}}{\partial y'} - \frac{\partial \mathcal{L}}{\partial y} = 0$$

Check: if $\mathcal{L}[y',y,x]=y'^2$ $\Rightarrow$ $\frac{d}{dx}(2y') - 0 = 0$ $\Rightarrow$ $\cancel{2}y''=0$

$\mathcal{L}$ is called Lagrangian

There are 2-D versions as well

to minimize $\iint \mathcal{L}\left[\frac{\partial \psi}{\partial x}, \frac{\partial \psi}{\partial y}, \psi, x, y\right] dx\, dy$

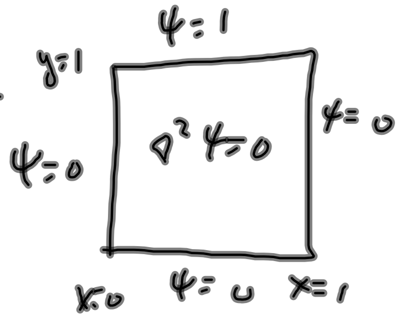
$\Rightarrow$ PDE: $\ldots \psi \ldots = 0$

to minimize $\iint \left(\psi_x^2 + \psi_y^2\right) dx\, dy$

$$\Rightarrow \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$$

(in Salikuri's table)

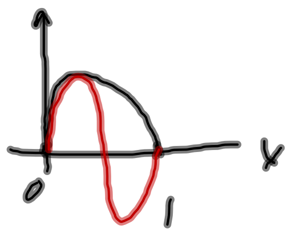
Example

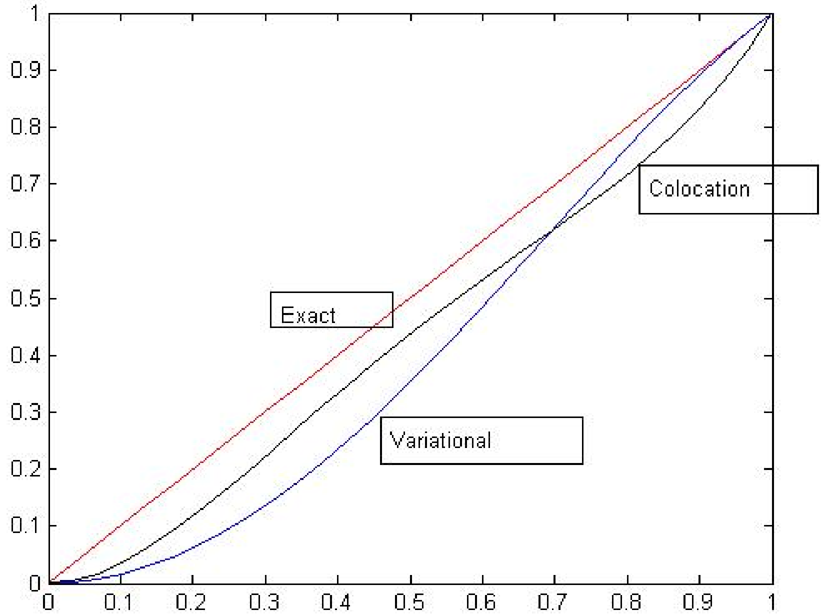


minimize $\iint (\psi_x^2 + \psi_y^2)\, dx\, dy$

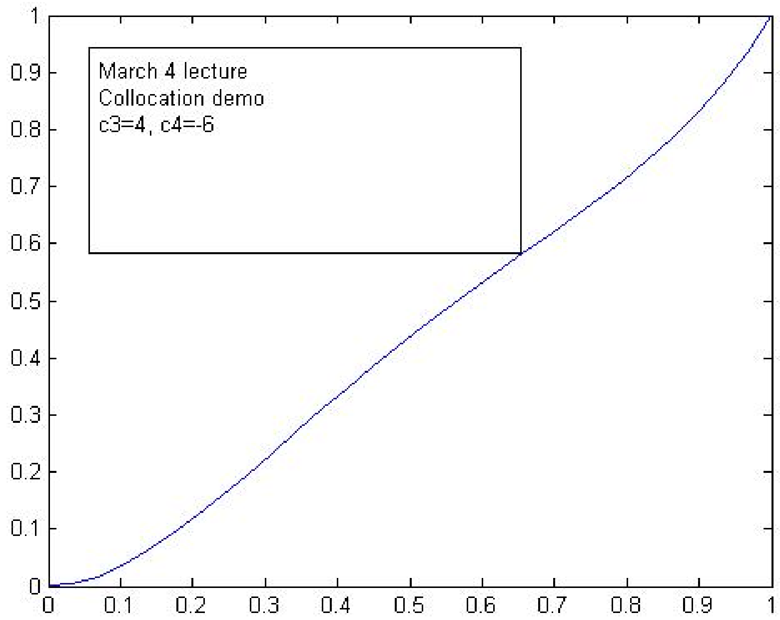
Basis expansion $c_1 (x)(x-1)\, y + c_2 \sin \pi x \sin (\;) y$

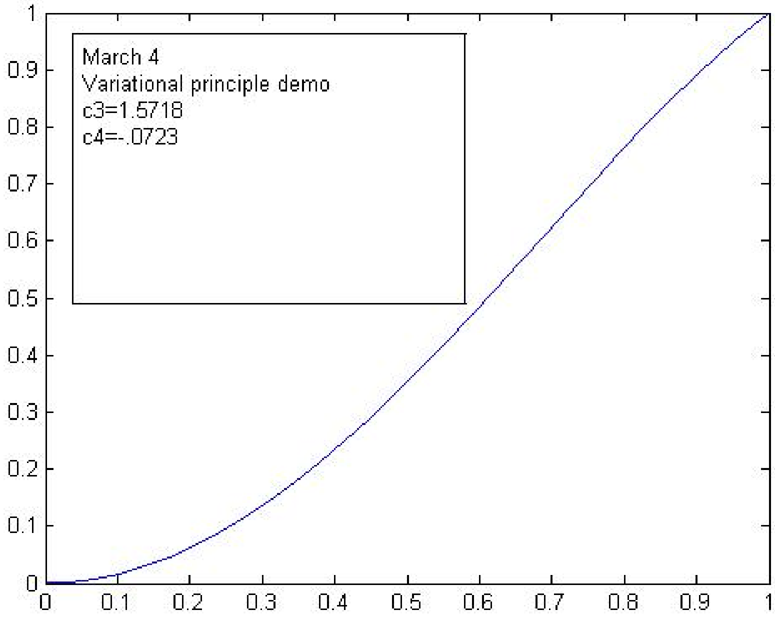
$+ c_3 \sin 2\pi x \sin (\;) y$





Exact
Colocation
Variational


March 4 lecture
Collocation demo
c3=4, c4=-6
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1


March 4
Variational principle demo
c3=1.5718
c4=-.0723
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

Reference: Sadiku p.257...

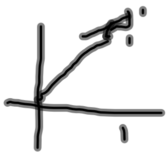
DE: $\phi'' + 4\phi - x^2 = 0$

BC.

$\phi(0) = 0$

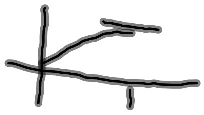
$\phi'(1) = 1$ ← nonhomog.

<u>Basis</u> $u_0(x) =$



$\phi = \cancel{x} u_0(x) + c_1 u_1 + c_2 u$

$u_{1,2,} =$



homog. BC.

$u_0(x) = x$

$u_n(x) = x^{n+1} - \underbrace{x^n \frac{n+1}{n}}_{\text{slope } n+1}$

at $x=1$ ↑ slope $n+1$

$u_1(x) = x^2 - x\,\frac{2}{1} = x(x-2)$

$u_2(x) = x^3 - x^2\,\frac{3}{2} = x^2(x - \frac{3}{2})$

approx $\phi \simeq x + c_1 x(x-2) + c_2 x^2 (x - 3/2)$

DE: $\phi'' + 4\phi = x^2$

Collocation: enforce the DE at $x = 1/3$, $x = 2/3$

$$c_1 = \frac{677}{548} \qquad c_2 = \frac{343}{548}$$

instead enforce at $x = \frac{1}{4}$ and $3/4$

$$c_1 = \frac{543}{412} \quad c_2 = \frac{288}{412}$$

Method of moments, weighted residual, subdomain method

$$\phi' + 4\phi = x^2$$

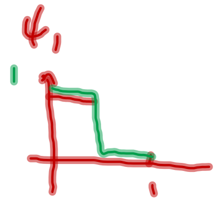
LHS RHS

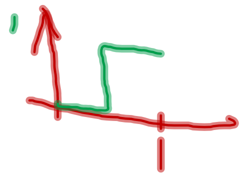
R = Residual = LHS − RHS

Pick 2 test fcns. ψ_1 & ψ_2

Enforce $\int R\,\psi_1\,dx = 0$

$\int R\,\psi_2\,dx = 0$





approx $\phi \simeq x + c_1 x(x-2) + c_2 x^2(x-3/2)$

DE: $\phi'' + 4\phi = x^2$

Residual = LHS − RHS = $0 + c_1 \cdot 2 + c_2 6x + c_2(2)(-3/2)$

$-x^2$

weight with ψ_1: $\int_0^{1/2} (\text{——})\,dx$

" with ψ_2: $\int_{1/2}^{1} (\text{——})\,dx$

$$\left.\begin{array}{l} \\ \\ \end{array}\right\} \quad c_1 = \frac{53}{48} \qquad c_2 = \frac{28}{38}$$

Least Squares method.

approx $\phi \simeq x + c_1 x(x-2) + c_2 x^2(x - 3/2)$

DE: $\phi'' + 4\phi = x^2$

minimize $\int_0^1 (\text{Residual})^2 dx$: $\frac{\partial}{\partial c_1} = 0$, $\frac{\partial}{\partial c_2} = 0$

$$\begin{bmatrix} - \\ - \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix} = \begin{Bmatrix} - \\ - \end{Bmatrix}$$

$c_1 = \frac{3826}{2842}$ $c_2 = \frac{2023}{2842}$

Galerkin method

$$u_0 = x$$
$$u_1 = x(x-2)$$
$$u_2 = x^2(x-3/2)$$

$$\phi = u_0 + c_1 u_1 + c_2 u_2$$

$$\phi'' + 4\phi = x^2$$

LHS RHS

R = Residual = LHS − RHS

Pick 2 test fcns. $\psi_1 = u_1 = x(x-2)$

$\psi_2 = u_2 = x^2(x-3/2)$

$$\int_0^1 R\psi_1 = 0$$
$$\int_0^1 R\psi_2 = 0$$

$$\begin{bmatrix} & \\ & \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \ \\ \ \end{bmatrix}$$

$$c_1 = \frac{694}{487} \qquad c_2 = \frac{301}{487}$$

Look at this formally:

D.E.: $\underbrace{\cdots\phi'' + \cdots\phi' + \cdots\phi + \cdots\phi \cdots}_{\text{LHS}} = g$ ← nonhomogeneity

$$\underbrace{\text{if linear D.E.,}}_{L\phi}$$

L ← operator

$$\phi'' + 4\phi = L\phi$$

$$L = \frac{d^2}{dx} + 4$$

DE $\quad L\phi = g \quad$ B.C.'s $\underbrace{}_{=0}$ $\underbrace{}_{=1 \;\; = \sin x}$

homog. $\qquad$ nonhomog.

approximate solution

$$\phi_{app} \simeq u_0 + c_1 u_1 + c_2 u_2 + \cdots + c_n u_n$$

u_0: satisfies all B.C.'s.

$u_1, u_2, \ldots, u_n$: satisfy homog. forms of B.C.'s. — basis functions

Plug in ϕ_{app}.

(No)

$$L\phi_{app} = g$$

$R_{residual} = R = L\phi_{app} - g$ ought to be zero.

To determine $c_1, \dots, c_n$; pick n test fcns. $\psi_1, \dots, \psi_n$, and integrate

$$\left.\begin{aligned} &\int R \cdot \psi_1 \, dx = 0 \\ &\int R \cdot \psi_2 \, dx = 0 \\ &\quad \vdots \\ &\int R \, \psi_n \, dx = 0 \end{aligned}\right\} \text{ n eqs.}$$

linear

$$\begin{bmatrix} \ \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} \vdots \end{bmatrix}$$

Considerations!

1. The basis fcns. $\{u_1 - u_n\}$ has to be rich enough that you believe they can approximate the solution.
2. To get high accuracy, you need lots of basis functions (n is large.)
3. The test fcns. have to cover the whole domain, not too smooth, lots of them.

4. Details

$$LHS = L\phi_{app} = L\{u_0 + c_1u_1 + c_2u_2 + \cdots + c_nu_n\}$$

$$= \underbrace{Lu_0 + c_1Lu_1}_{} + c_2Lu_2 + \ldots c_nLu_n$$

Residual $= LHS - RHS = \sum_{j=0}^{n} c_j u_j - g$

$(c_0 = 1)$

Enforce:

$$\int (LHS - RHS)\psi_k\,dx = 0, \quad K = 1, \ldots, n$$

$$\int \underset{L}{u_0}\psi_k\,dx + \sum_{j=1}^{n} c_j \int \underset{L}{u_j}\psi_k\,dx = \int g\psi_k\,dx$$

$$\cancel{\int u_0\psi_k\,dx} + \sum_{j=1}^{n} c_j \int \underset{L}{u_j}\psi_k\,dx = \int g\psi_k\,dx - \int \underset{L}{u_0}\psi_k\,dx$$

insert L

L ...

$$\begin{bmatrix} \int u_1\psi_1 & \int u_2\psi_1 & & \int u_n\psi_1 \\ \int u_1\psi_2 & \int u_2\psi_2 & & \int u_n\psi_2 \\ & & & \\ \int u_1\psi_n & & & \int u_n\psi_n \end{bmatrix} \begin{bmatrix} c_1 \\ \\ \\ c_n \end{bmatrix} = \begin{bmatrix} \int (g - Lu_0)\psi_1 \\ \int (g - Lu_0)\psi_2 \\ \vdots \\ \int (g - Lu_0)\psi_n \end{bmatrix}$$

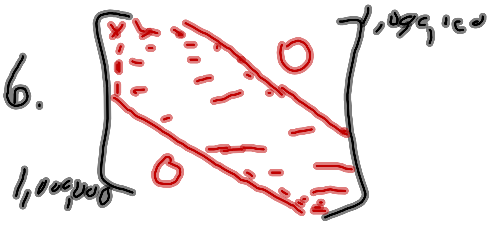
$$\begin{bmatrix} \int L u_1 \psi_1 & \int L u_2 \psi_1 & \cdots & \int L u_n \psi_1 \\ \vdots & & & \vdots \\ \int L u_1 \psi_n & & & \int \cdots \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} \int (g - L u_0) \psi_1 \\ \vdots \\ \int (g - L u_0) \psi_n \end{bmatrix}$$

"Matrix"

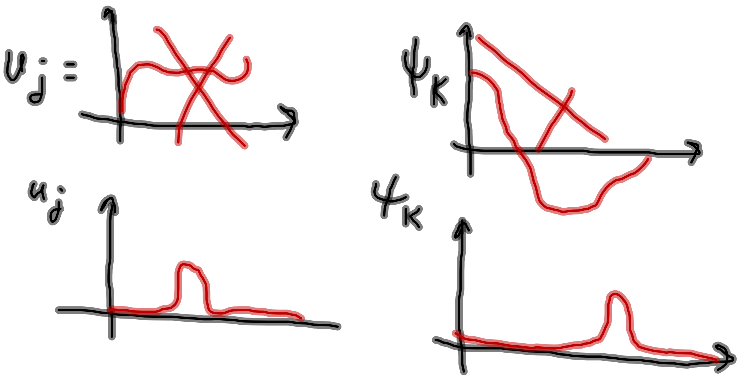
"Stiffness"

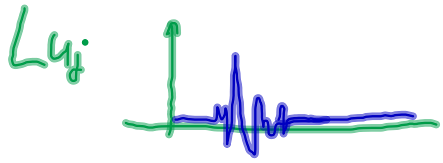
(Considerations)

5. Try to minimize the work involved in evaluating $\int L u_j \psi_k \, dx$

6.  Try to make matrix sparse.

Common solution:





most of the time, $\int L u_j \psi_k = 0$

because their supports don't overlap. $\Rightarrow$ Sparseness.

When they do overlap, $\int_{\text{domain}} L u_j \psi_k = \int_{\text{subdomain}} L u_j \psi_k$, less work.

Use local functions for basis & test set.

General Structure of Method of Moments.

Have DE $\quad a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x) \qquad Ly = g$

BC's $\quad \dots y(0) \dots y'(0) \dots = \# \qquad L^{(1)}_{BC}\, y = \# = 0$

$\quad \dots \dots = \# \qquad L^{(2)}_{BC}\, y = \# = 0$

Homogeneous BC's

Trial soln: $y_{app} = u_0 + c_1 u_1 + c_2 u_2 + \dots + c_n u_n$

u_0 satisfies non-homog. BC's

$u_1, u_2, \dots, u_n$ satisfy the homog. form of the BC's. "basis functions"

$$= u_0 + \sum_{j=1}^{n} c_j u_j(x)$$

Try to enforce

$$L\left\{u_0+\sum_1^n c_j u_j\right\}=g \quad \text{at every } x.$$

We ~~can~~ only have n c's to adjust.

Note $L y_{exact} - g \equiv 0$

So $$\int (L y_{exact} - g)\,\psi\, dx = 0$$

any fcn.

So to get n equations (determining the n c's), pick n different functions (test functions) and enforce

$$\int \left\{ L\left(u_0 + \sum_1^n c_i u_n\right) - g \right\} \psi_k \, dx = 0$$

What do the n equations look like?

$$\int L u_0 \, \psi_1 + \sum c_j \int L u_j \psi_1 - \int g \psi_1 = 0$$

$$\int L u_0 \psi_2 + \sum c_j \int L u_j \psi_2 - \int g \psi_2 = 0$$

what do the n equations look like?

$$\int L u_0 \psi_1 + \sum c_j \int L u_j \psi_1 - \int g \psi_1 = 0$$

$$\int L u_0 \psi_2 + \sum c_j \int L u_j \psi_2 - \int g \psi_2 = 0$$

$$\vdots$$

$$- - - - - - - - \psi_n = 0$$

$$c_1 \int L u_1 \psi_1 + c_2 \int L u_2 \psi_1 + \cdots\cdots = \int (-L u_0 + g) \psi_1$$

$$\vdots \quad \psi_2$$

$$\vdots \quad \psi_n$$

$$\begin{bmatrix} \int L u_1 \psi_1 & \int L u_2 \psi_1 & & \\ & \ddots & & \\ & & \ddots & \\ & & & \int L u_n \psi_n \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} \int (-L u_0 + g) \psi_1 \\ \int (-L u_0 + g) \psi_2 \\ \vdots \\ \psi_n \end{bmatrix}$$

$$\begin{bmatrix} \int L u_1 \psi_1 & \int L u_2 \psi_1 & & \\ & \cdot & \cdot & \\ & & & \cdot \\ & & & \int L u_n \psi_n \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} \int (-L u_0 + g)\psi_1 \\ \int (-L u_0 + g)\psi_1 \\ \vdots \\ \psi_n \end{bmatrix}$$

① This formulation is full of $\int f(x)\, g(x)\, dx$.

Use "scalar product" notation

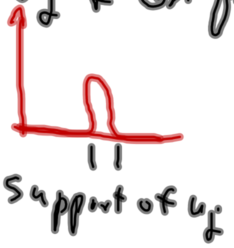
So $\int L u_1 \psi_1 dx = (L u_1, \psi_1)$ $\qquad \int f g = (f, g)$

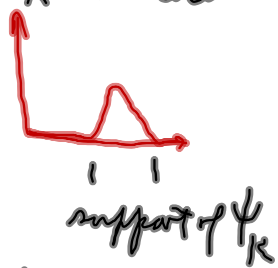
If have complex function then $(f, g) = \int f \bar{g}$

$$\begin{bmatrix} \int Lu_1\psi_1 & \int Lu_2\psi_1 & & \\ & \ddots & & \\ & & & \int Lu_n\psi_n \end{bmatrix}\begin{bmatrix} c_1\\ \vdots \\ c_n\end{bmatrix} = \begin{bmatrix}\int(-Lu_0+g)\psi_1 \\ \int(-Lu_0+g)\psi_2 \\ \vdots \\ \psi_n\end{bmatrix}$$

Lots of integrals (Lu_j, ψ_k)

We choose basis functions u_j & test functions ψ_k so that



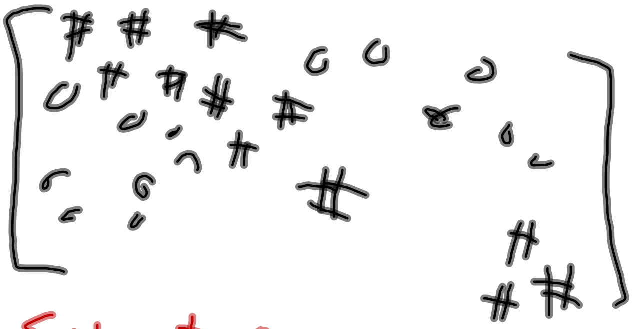


If the two supports don't overlap, $\int u_j \psi_k = 0$

$Ly = a_2 y'' + a_1 y' + a_0 y$ $\quad \int L u_j \psi_k = 0$

$$\begin{bmatrix} \int L u_1 \psi_1 & \int L u_2 \psi_1 & & \\ & & \ddots & \\ & & & \int L u_n \psi_n \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} \int(-Lu_0+g)\psi_1 \\ \int(-Lu_0+g)\psi_2 \\ \vdots \\ \psi_n \end{bmatrix}$$

Looks like

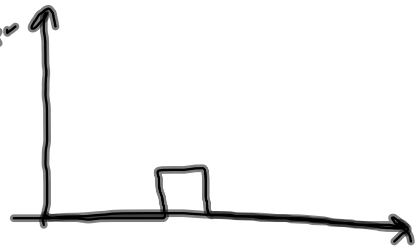


"sparse", easier for computer to handle

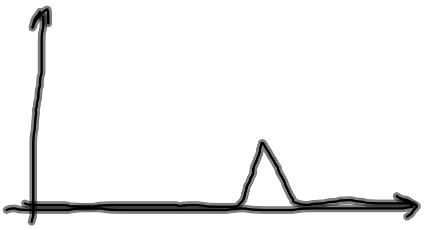
So: get sparseness by using local basis fns & test fcns.

Local functions.

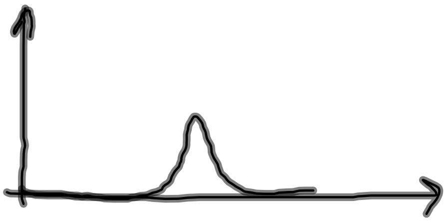
Rectangular pulse



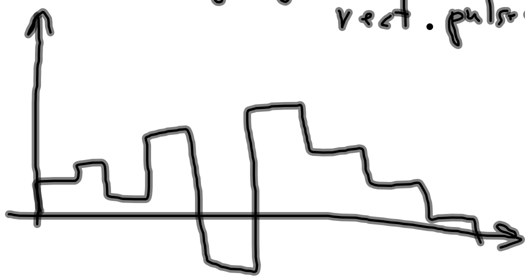
Triangular pulse



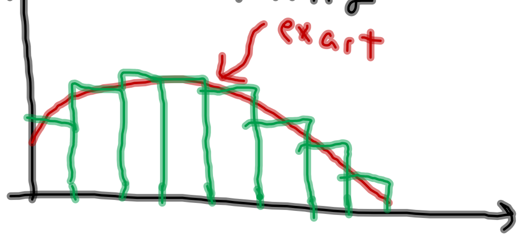
smoother pulses "splines"



Resulting approximation $\sum c_j u_j$ for rect. pulses.

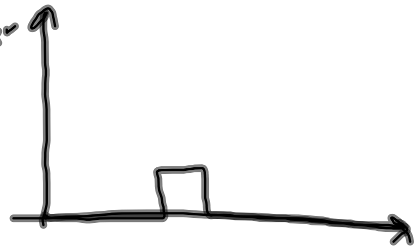


more realistically

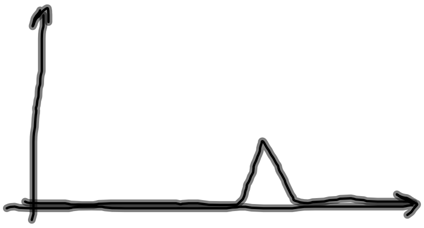


Local functions.

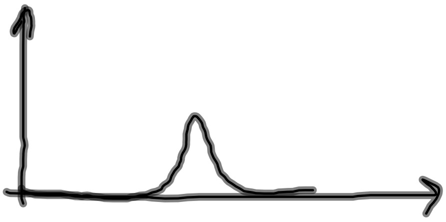
Rectangular pulse



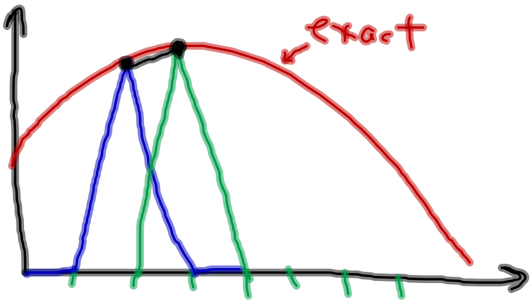
Triangular pulse

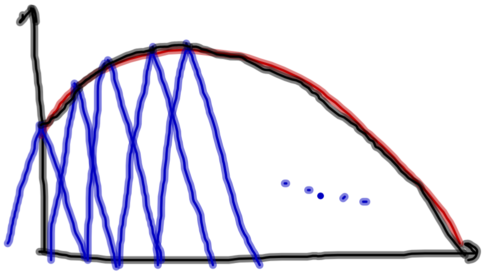


smoother pulses "splines"



Resulting approximation, triangular pulses $\sum c_j u_j$





Triangular pulses give you closer approximation than rectangular; (probably) quadratic splines are better still.

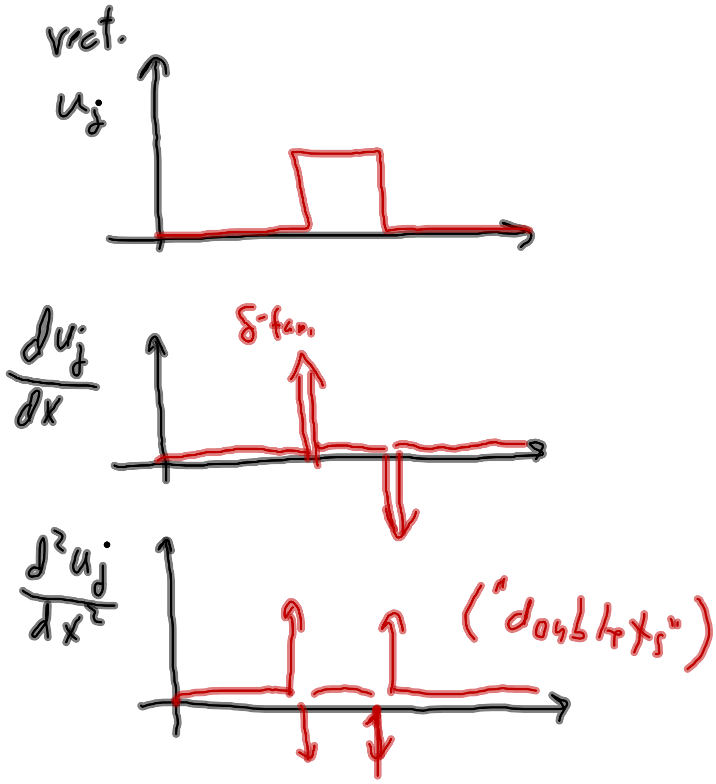
If you need to know the derivative of the solution y_{exact}, the rectangular pulses are useless. To approximate higher derivatives, you need smoother pulses.

$$L = a_2(x)\frac{d^2}{dx} + a_1(x)\frac{d}{dx} + a_0(x) \qquad L y_{exact} = g$$

$$\begin{bmatrix} & & \\ & (Lu_j, \psi_k) & \\ & & \end{bmatrix}$$

What is Lu_j for a pulse ?

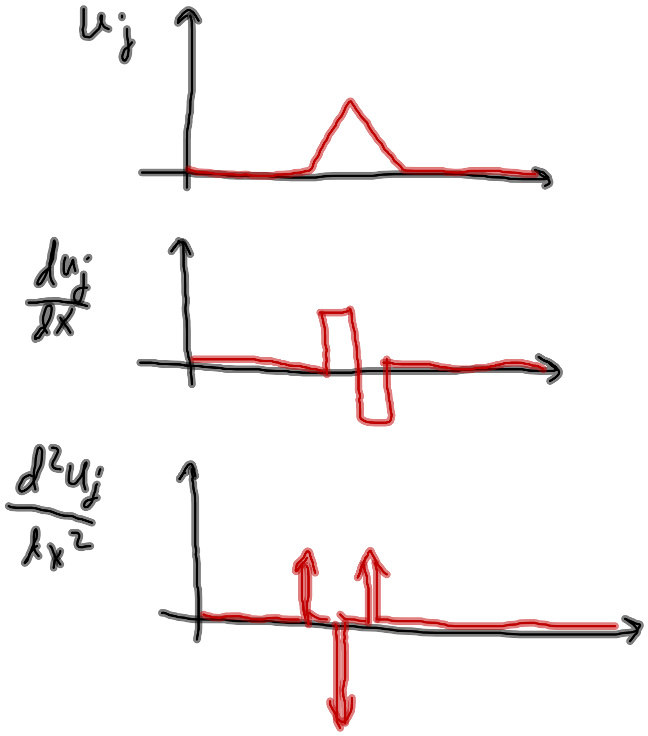
$$L = a_2(x)\frac{d^2}{dx} + a_1(x)\frac{d}{dx} + a_0(x) \qquad L\,y_{exact} = g$$



(Try to avoid)

$$L = a_2(x)\frac{d^2}{dx} + a_1(x)\frac{d}{dx} + a_0(x)$$

$$L\, y_{exact} = g$$



Try to avoid this.

Take a new tack to solve the DE.

original DE $\quad Ly = g \qquad a_2 y'' + a_1 y' + a_0 y = g$

weakened our requirements $\quad (Ly - g, \psi) = 0$ "strong form of DE"

strong form: $\displaystyle\int_a^b Ly\,\psi = \int g\psi$

$$\int_a^b (a_2 y'' + a_1 y' + a_0 y)\psi = \int g\psi)$$

integrate by parts on y'' $\quad \displaystyle\int_a^b (a_2\psi)\frac{d}{dy}y' + (\text{rest}) = \text{--}$

$\int u\,dv = -\int v\,du + uv\Big|$

$\psi)$

$$\underbrace{y' a_2 \psi\Big|_{BC}}_{\text{suppose BC make this } = 0.} \quad -\int_a^b y' \frac{d}{dx}(a_2\psi) + \text{rest}$$

suppose BC make this = 0.

The new substitute for the original DE & the strong form is $\quad \displaystyle\int_a^b -y'(a_2\psi)' + (a_1 y' + a_0 y)\psi = \int g$

The weak form of the DE.

$$\int_a^b -y'(a_2\psi)' + (a_1 y' + a_0 y)\psi = \int g \qquad \text{weak form}$$

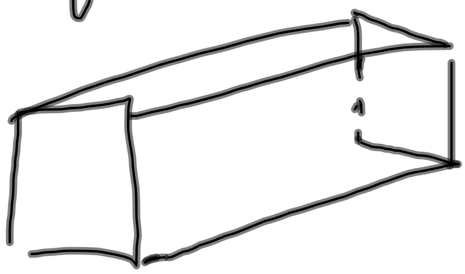
$$\int_a^b -a_2 y'\psi' - a_2' y'\psi + a_1 y'\psi + a_0 y\psi = \int g\psi$$

Play one more hand: take $\frac{d}{dx}$ ~~...~~ of y', get y.
Assume we can throw away the Boundary terms

$$\int +(a_2\psi')' y + (a_2'\psi)' y + (a_1\psi)' y + a_0 y\psi = \int g\psi$$

"Feeble form"

Assignment #9



Recall

$B_{inside} =$ (const) $\cdot \vec{K}$

TE mode: $E_z = 0$

$A_z = \boxed{?} B_z$

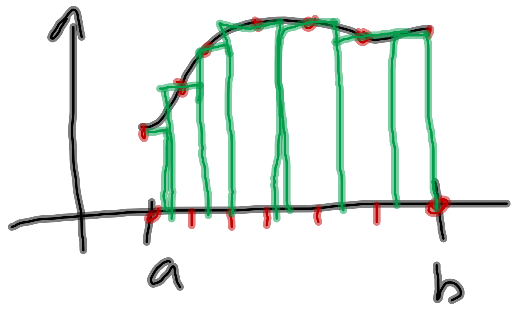
BC $\frac{\partial A_z}{\partial n} = 0$

$E = \nabla \cdots A_z$

$B_y, B_x = \nabla \cdots A_z$

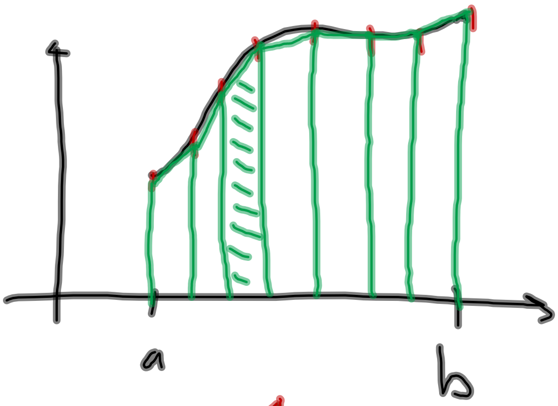
$\nabla^2 A_z + (\lambda_1) A_z = 0$

Numerical Integration.



rectangular

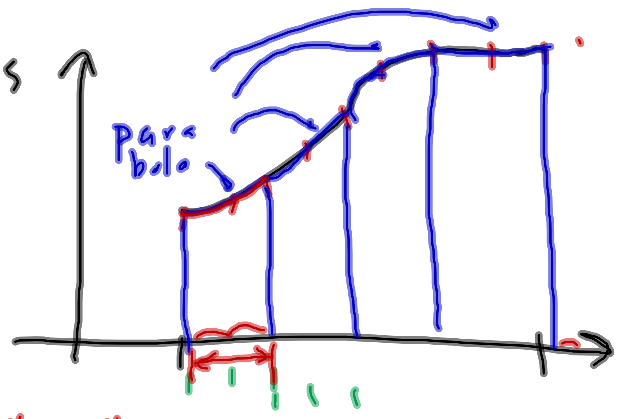
Trapezoidal



area is the same as — average of left-side & right-side height

$$\text{area} = \text{base} \times \frac{f(\text{left } x) + f(\text{right } x)}{2}$$

Simpson's Rule



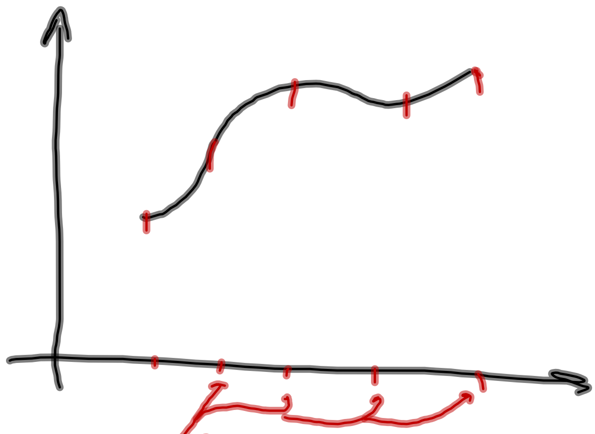
$$\text{"base"} \cdot \frac{f(\text{left}) + 4f(\text{center}) + f(\text{right})}{6}$$

"base" = $2\Delta x$

1 4 2 4 2 4 2 4 1

If "base" = Δx, use 3 instead of 6

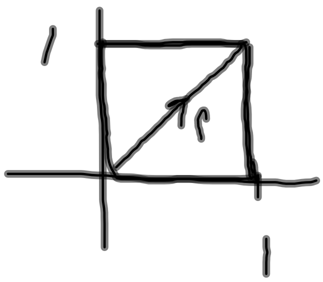
Gaussian quadrature



If you ~~?~~ can pick the x-coords of the data

$$\int_0^1 \underbrace{\int_0^x x^2 y^2 \, dy}_{} \, dx$$

$$\int_\Gamma f(x,y)\,dl = \int_\Gamma x^2 y^2 \, dl$$



on Γ: $y = x$

$$dl = \sqrt{dx^2 + dy^2}$$

$$= \sqrt{2\,dx^2}$$

$$= \sqrt{2}\,dx$$

$$\int_\Gamma \cancel{\infty} = \int_{x=0}^{x=1} x^2 x^2 \sqrt{2}\,dx$$

DE: $L\psi = g$ BC $M\psi = h$

MoM basic strategy

① approximate $\psi \simeq \sum_n c_n u_n$

basis fn's.

② Enforce weak forms of the eqs:

$$\langle L\psi - g, w_j \rangle = 0 \qquad \langle M\psi - h, w_j \rangle = 0$$

for a selected set of test fcn,

Good choices for basis & test functions are ones whose supports seldom overlap:

because then

$$\langle L \cancel{u} - g, w_j \rangle$$

$$\langle L \sum c_n u_n - g, w_j \rangle = 0$$ most of these integrals are zero.

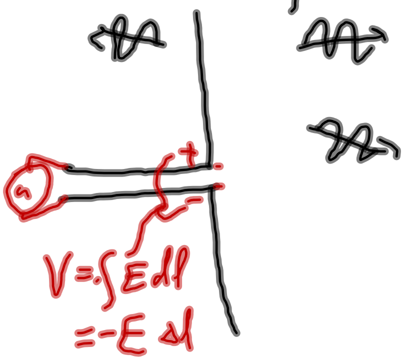
$\Rightarrow$ "SPARSE" coefficient matrix in the equations.

Will come back to this philosophy when we look at finite elements (Chpt. 6).

Change approach (Chpt. 5).

EM problem:

Radiating antenna

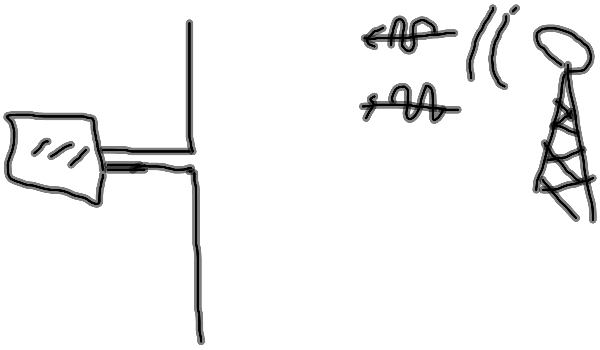


$V = \int E\,dl$

$= -E\,\Delta l$

Given: voltage driving the antenna

Receiving antenna



Given: $\vec{E}_{\text{incoming}}$

In each case, instead of trying to find the vector potential $\vec{A}(x,y,z;\omega) = \sum_{\text{lots}} c_n \vec{u}_n(-)$ over all space,

instead approximate the current on the wire $\vec{j} = \sum_{\substack{\text{many}\\\text{fewer}}} c_n \vec{u}_n(x,y,z)$,

$\Longrightarrow$ then compute $\vec{A}$ in terms of $\vec{j}$,

then enforce forms of

$\vec{E}(\text{antenna}) = \text{given}$, $\vec{E}_{\text{tan, wire}} = 0$, $\vec{E}_{\text{total}} = \vec{E}_{\text{inc}} + \vec{E}_{\text{computed from } \vec{j}}$.

$\Rightarrow$ Then compute $\vec{A}$ in terms of $\vec{j}$,

This is very difficult, but it is possible.

Sample: $\vec{j}_{given}$ is infinitesimal:

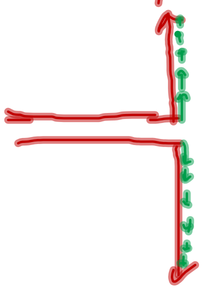
$$\vec{j} = \vec{I}\,\delta(x-x_0)\,\delta(y-y_0)\,\delta(z-z_0)\,e^{i\omega t}$$

↑

Then $\vec{A}$ can be found. Chpt 5, p. 275

These solutions are called "Green's fcns."

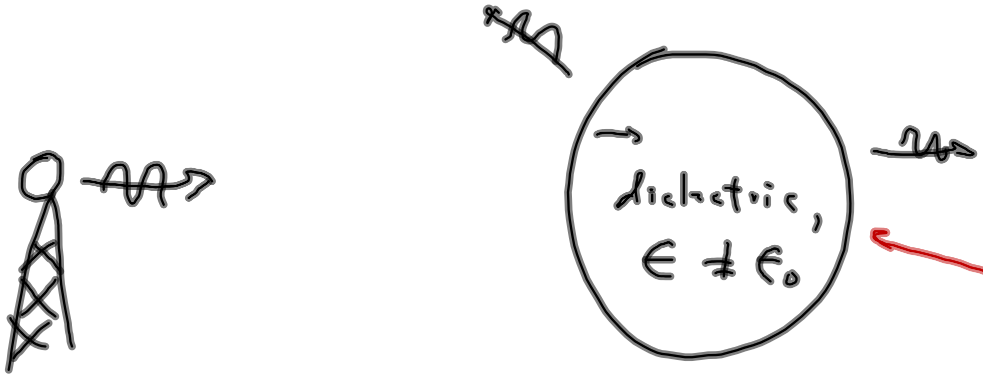
What do you do if you have a finite current density?



$$\vec{j}_{actual} = \sum c_n \, \vec{j}_{delta\text{-}fcns}$$

$$\vec{A}_{actual} = \sum c_n \,(\text{Green's functions})$$

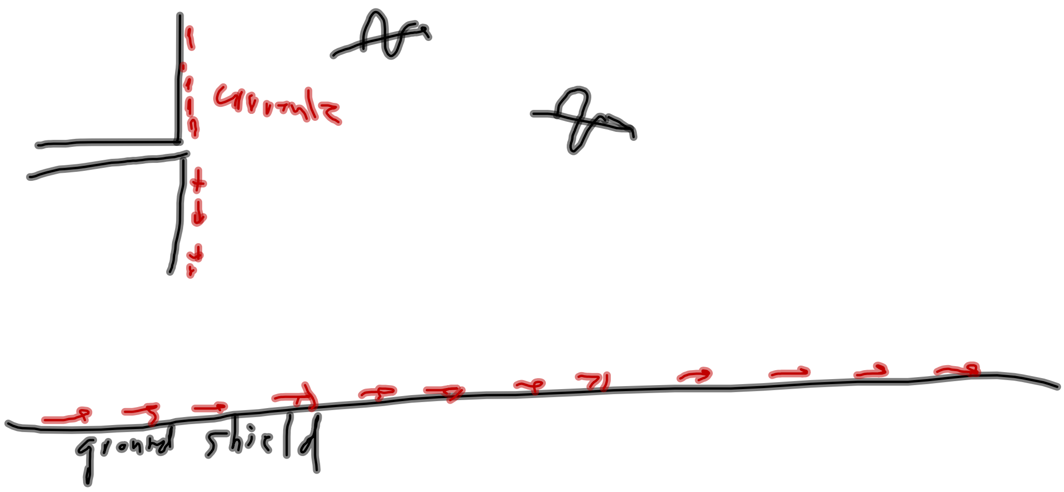
Why don't we use this approach all the time?



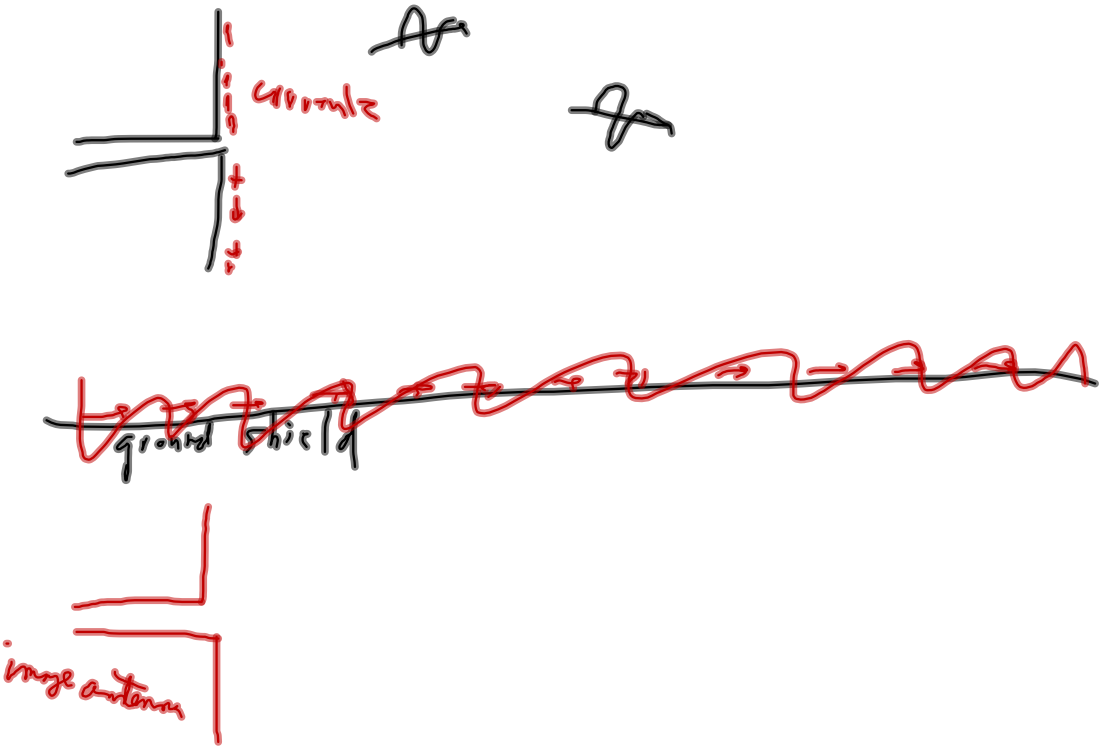
We could compute $\vec{A}$ if we knew (or modeled) all the current that give rise to it ($\vec{J}$)

But there are dielectric dipole currents all through the dielectric!

Another case



Another case

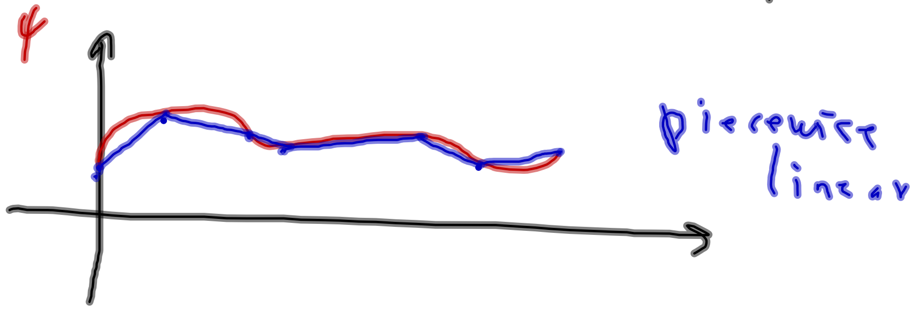


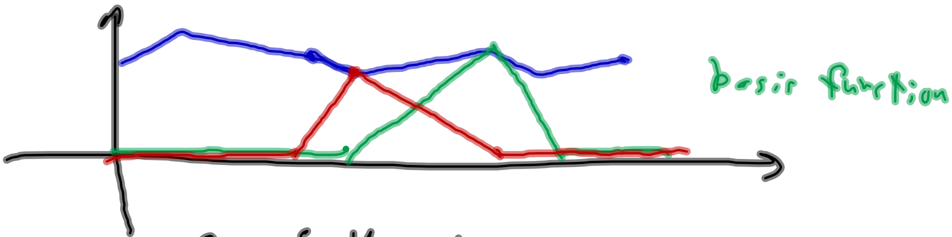
Thats chapter 5!

Chapter 6. Finite Elements.

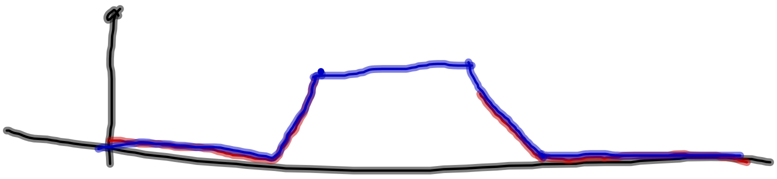
" " are particular choices for the basis functions, used in MoM.

One dimensional finite elements.





sum of these ?



So the piecewise linear approximation is a sum of  hat-functions,

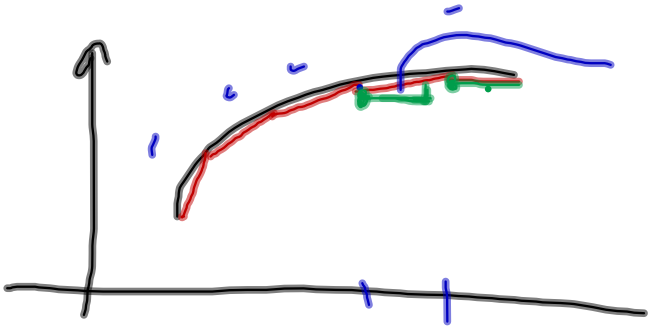
$$f_{pie\text{-}lin} = \sum_{n} c_n \, u_n(x)$$

hat function with peak at x_n

Finite Element Issues

1. The basis functions are convenient for the computation of the Lagrangian to be minimized or the weak form of the differential equations; and at the same time general enough to achieve close approximation of the solution.
2. The geometric region is triangulated (or rectangulated) without restriction to accommodate irregularities (corners, curved boundaries, …) and rapidly changing physical features (variable or discontinuous permittivities, conductances…).
3. The basis functions are polynomials of low degree.
4. Higher accuracy is obtained, not by introducing more basis functions, but by refining the triangulation and using the same polynomials.
5. Much of the work of specifying and refining the triangulations can be done by the computer.
6. Clever parametrizations of the basis functions ensure heir continuity from triangle to triangle, ***AND*** give physical significance to the mathematical coefficients in the approximation

Finite elements in 1-D.

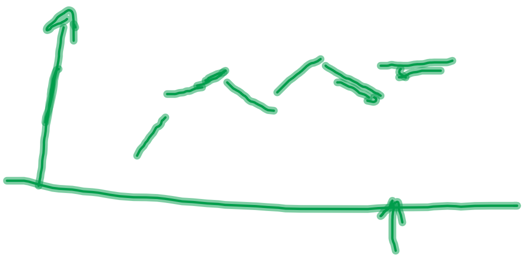


straight line

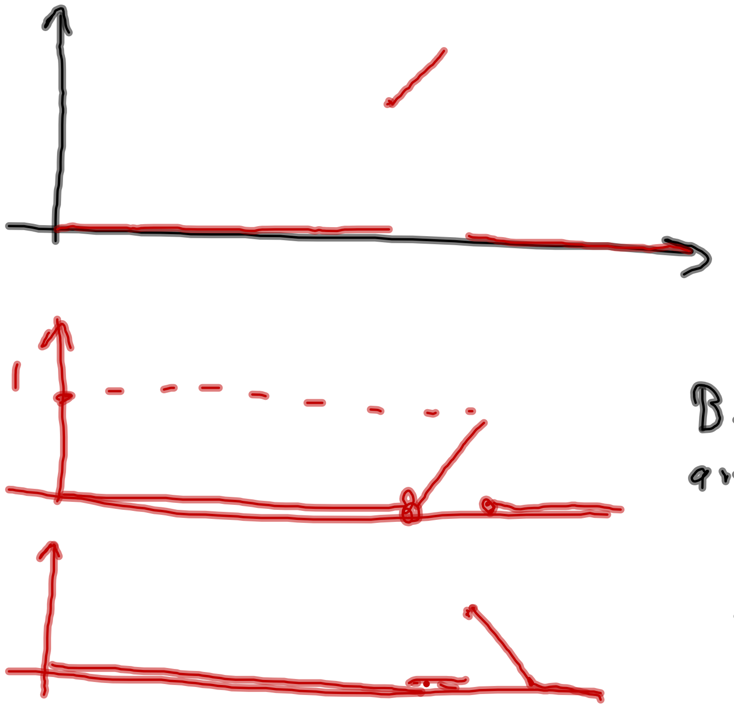
give a point &

a slope.

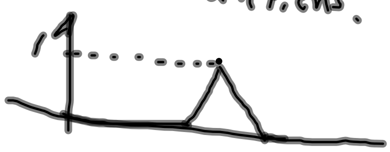
(x_1, y_1), m

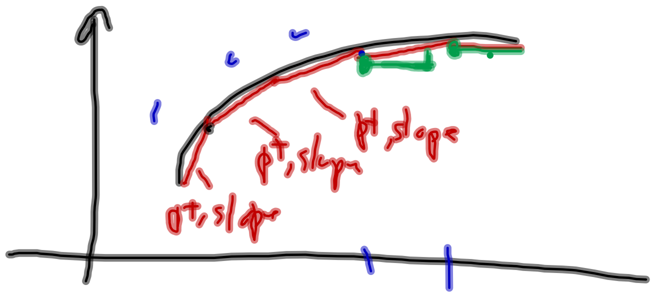


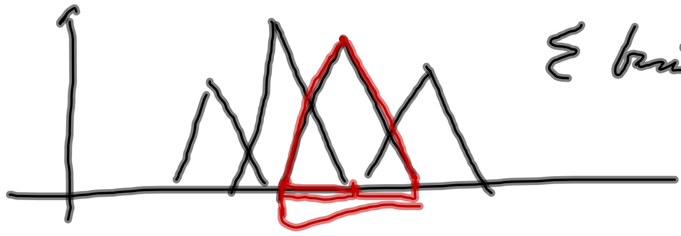
To ensure -continuity



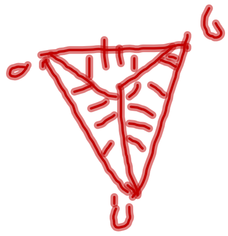
Basis functions are hat functions.



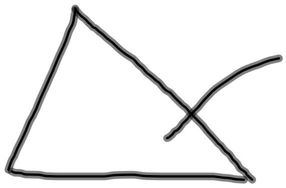




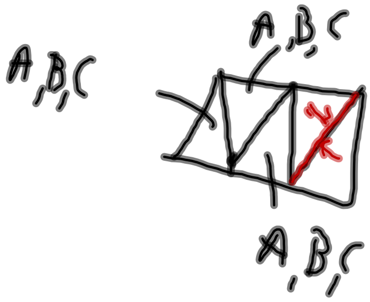
Σ basis fcns · values at peaks.



2-D FE's <u>linear</u>



$$f(x,y) = Ax + By + C$$



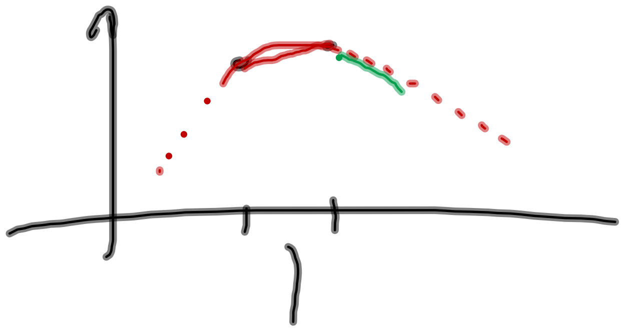
1-D FE quadratic

In EM
$$\frac{\partial \rho}{\partial t} = \nabla \cdot \vec{J}$$

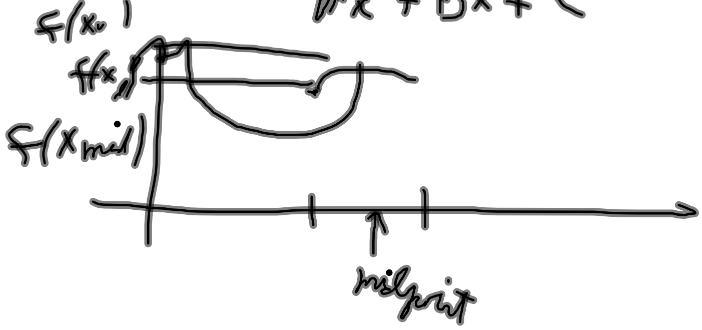
at DC $\Longrightarrow \nabla \cdot \vec{J} = 0$

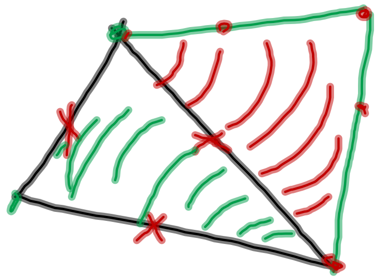
$$\underbrace{\frac{\partial}{\partial x} j_x}_{A} + \underbrace{\frac{\partial}{\partial y} j_y}_{B} + \underbrace{\frac{\partial}{\partial z} j_z}_{C} = 0$$

$j_x = Ax + By + Cz + D$



$Ax^2 + Bx + C$

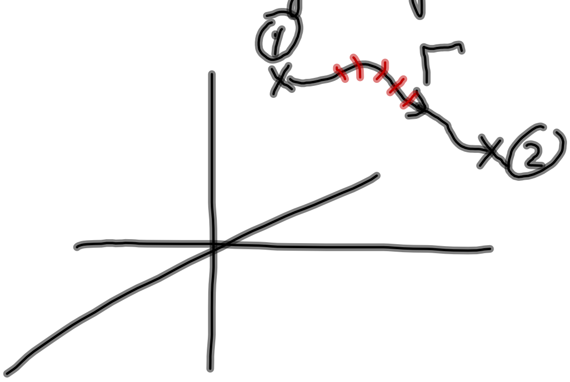




$$Ax + By + C$$
$$+ Exy + Fx^2 + Gy^2$$

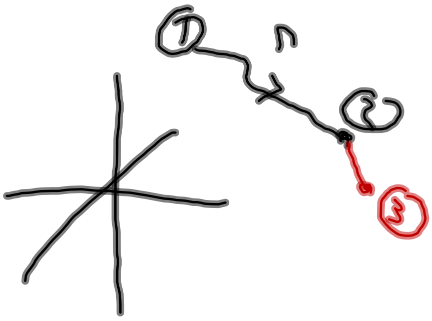
The notion of voltage.

Voltage ≡ potential ≡ potential energy per unit charge.



Work in going from 1 to 2 along path Γ

is $\int \underbrace{q\vec{E}}_{\vec{F}} \cdot d\vec{r} = -$ potential energy



÷ q potential

∇ potential $= -\vec{E}$

<u>Math</u> $\vec{\nabla} \times \vec{\nabla}(\text{pot.}) = \vec{0}$

$$\vec{\nabla} \times \vec{E} = (-)\vec{0} = \vec{0}$$

<u>BUT</u> $\vec{\nabla} \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t} \neq 0$

The Flaw $\left(\nabla \times \vec{E} = 0 \,,\ \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \right)$

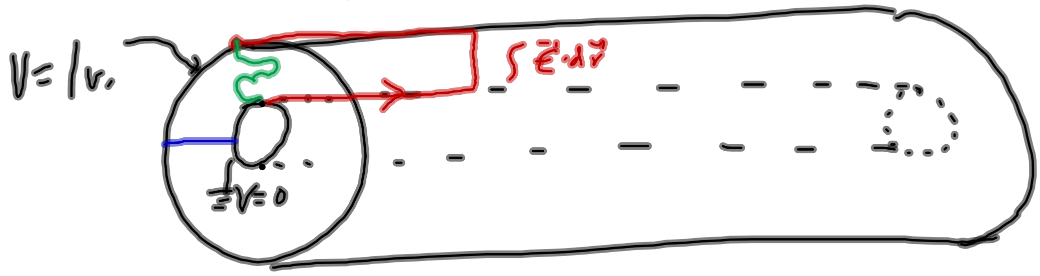
is that

$$\int_{①}^{②} \vec{E}\, d\vec{r} = -\text{"potential"}$$

depends on the path chosen to get from ① to ②,

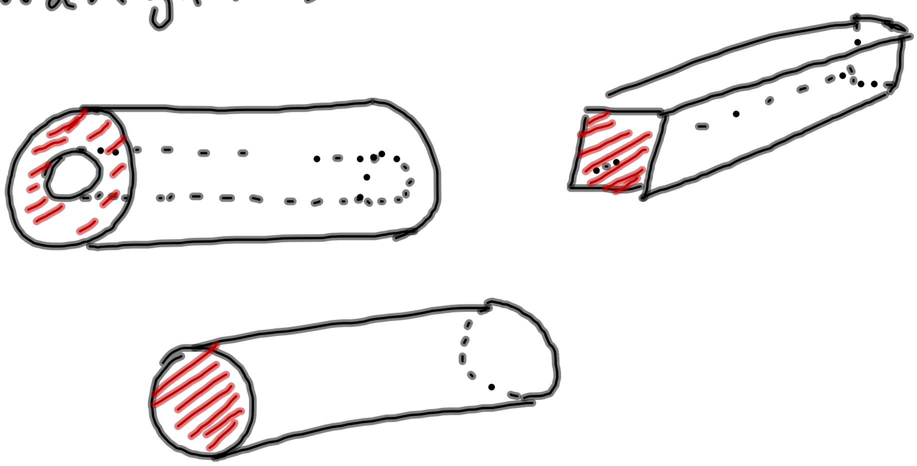
except when $\frac{\partial \vec{B}}{\partial t} = 0$.

What do microwave engineers mean, when they say "voltage"?



$\int_{inner}^{outer} \vec{E}\cdot d\vec{r}$ changes with path!

Waveguides



Across a face, the $\vec{E}$ & $\vec{B}$ are complicated.

"Voltage" is even shakier.

Fact: $\vec{E}(x,y,z,t)$, $\vec{B}(x,y,z,t)$, $V(x,y,z,t)$

$$= \text{Complicated}(x,y) \cdot e^{i(kz-\omega t)}$$

This factor governs how the fields propagate down the line.

Note: the relation k & ω (dispersion relation) is governed by "Complicated(x,y)" part.

Lumped parameters.

$\vec{E}(x,y,z,t)$, $\vec{B}(x,y,z,t)$ complicated

$\Longrightarrow$ compute Voltage, Current

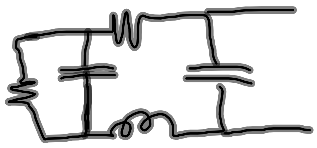
If V, I are defined in a convenient manner

you often get simple relationships

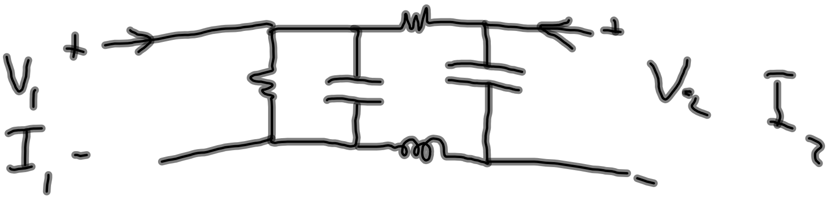
$\frac{V}{I}$ = const (complex) if $V = R_{eq} I$ "resistor"

$V = i\omega L I$ "inductor"

$V = \frac{1}{i\omega C} I$ "capacitor"



$\frac{V}{I}$ = complex.



"Two-port"

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} - & - \\ - & - \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

At microwave frequencie