1-dimensional Gaussian distribution $N(\mu, \sigma)$

$$\frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

2-D Gaussian p. 191

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_1}{\sigma_1}\right)^2 + \left(\frac{y-\mu_2}{\sigma_2}\right)^2 \cdots\right.}$$

$$\left.\cdots - \frac{2\rho(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2}\right]$$

$$= (\text{const})\, e^{x^2,\, y^2,\, xy,\, x,\, y,\, \text{const}}$$

1-D

$$\text{Prob}\,(a < X < b) = \int_a^b f_X(x)\,dx$$

2-D

Joint distribution function

$$\text{Prob}\,(a < x < b \text{ and } c < y < d) = \int_a^b \int_c^d f_{X,Y}(x,y)\,dy\,dx$$

$$\mathcal{E}(X) = \mu_1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\, f_{XY}(x,y)\,dy\,dx$$

$$\mathcal{E}(Y) = \mu_2 = \iint y\, f_{XY}$$

$$\mathcal{E}(X^2) = \iint x^2 f_{XY} \cdots$$

$$\sigma_1^2 = \mathcal{E}\left[(X-\mu_1)^2\right] = \mathcal{E}\left[X^2\right] - \mu_1^2$$

Marginal Distributions

Prob (a < x < b and y is anything)

$$= \int_{x=a}^{b} \int_{y=-\infty}^{\infty} f_{X,Y}(x,y)\,dy\,dx$$

marginal on X : $f_X(x) = \int_{y=-\infty}^{\infty} f_{X,Y}(x,y)\,dy$

Fact The marginal density function for the 2-D Gaussian is the 1-D Gaussian

$$\int_{-\infty}^{\infty} \frac{1}{2\pi\sigma_1\sigma_2} e^{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_1}{\sigma_1}\right)^2 + \left(\frac{y-\mu_2}{\sigma_2}\right)^2 - 2\rho\frac{(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2}\right]} dy$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma_1} e^{-\frac{1}{2}\left(\frac{x-\mu_1}{\sigma_1}\right)^2}$$

~~Conditional Prob~~

Incidental Remark :

You can write the bilinear form, in the 2-D exponential, in a matrix form :

$$- \begin{bmatrix} x-\mu_1, & y-\mu_2 \end{bmatrix} \begin{bmatrix} X & X \\ X & X \end{bmatrix} \begin{bmatrix} x-\mu_1 \\ y-\mu_2 \end{bmatrix}$$

Conditional probability.

$P(A, \text{given } B)$

If A & B are independent, ← This is the same as $P(A)$

Basic rule:

given

$$P(B)\,P(A|B) = P(A+B) = P(A)\,P(B|A)$$

$$P(B|A) = \frac{P(A+B)}{P(A)}$$

$$\text{conditional} = \frac{\text{joint}}{\text{marginal}}$$

$$P(B|A) = P(A|B)\,P(B) / P(A)$$

Bayes' Thm.

and only if

IF A + B are independent,

$$P(A+B) = P(A)\,P(B)$$

Fact: for density functions,

$$\text{conditional} = \frac{\text{joint}}{\text{marginal}}$$

~~Prob~~

$$\text{Prob}(x \le X \le x+dx, \underbrace{\text{given that } y \le Y \le y+dy}_{\text{"}Y=y\text{"}}) = \frac{\text{Prob}(x \le X \le x+dx \text{ and } y \le Y \le y+dy)}{\text{Prob}(y \le Y \le y+dy)}$$

corrected at end of lecture

conditional

$$f_{X|Y}(x;y)\,dx = \frac{f_{XY}(x,y)\,dx\,dy}{f_Y(y)\,dy}$$

$$f_{X|Y}(x; \text{"}Y=y\text{"})\,dx$$

(Trust me)

$$f_{X|Y}(x;y) = \frac{f_{XY}(x,y)}{f_Y(y)}$$

$$f_{X|Y}(x;y) = \frac{f_{XY}(x,y)}{f_Y(y)}$$

For the Gaussian

LHS to be determined } will turn out to be Gaussian in x

$$N\left(\mu_1 + \rho\frac{\sigma_1}{\sigma_2}(y-\mu_2),\ \sigma_1\sqrt{1-\rho^2}\right)$$

mean

p. 193, 4

$$f_{X|Y}(x;y) = \frac{f_{XY}(x,y)}{f_Y(y)}$$

$$\frac{1}{\sqrt{2\pi}\,\sigma_1\sqrt{1-\rho^2}}\, e^{-\frac{1}{2}\frac{\left[x-\mu_1-\rho\frac{\sigma_1}{\sigma_2}(y-\mu_2)\right]^2}{\sigma_1^2(1-\rho^2)}} = \frac{\left[\frac{1}{2\pi\sigma_1\sigma_2}\, e^{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_1}{\sigma_1}\right)^2+\left(\frac{y-\mu_2}{\sigma_2}\right)^2-2\rho\left(\frac{x-\mu_1}{\sigma_1}\right)\left(\frac{y-\mu_2}{\sigma_2}\right)\right]}\right]}{\int_{x=-\infty}^{\infty}(\text{above})\,dx}$$

NOTE: If $\rho = 0$ X & Y are independent

because

$$f_{XY} = \frac{1}{2\pi\sigma_1\sigma_2}\, e^{-\frac{1}{2}\left(\frac{x-\mu_1}{\sigma_1}\right)^2}\, e^{-\frac{1}{2}\left(\frac{y-\mu_2}{\sigma_2}\right)^2}$$