Goal: have measured $x(t)$, one of the possible outcomes of $X(t)$, want to use time averages to estimated

$$R_X(\tau) = E\left[X(t)\, X(t+\tau) \right]$$

(autocorrelation fcn.)

Use, in theory,
$$\lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, x(t+\tau)\, dt \; ;$$

MUST COMPUTE THIS LONG AVERAGE, FOR EVERY VALUE OF τ.

TOO MUCH WORK.

TRY AN ALTERNATIVE COMPUTING SCHEME:

Fourier transform

$$\tilde{X}(f) = \int_{-\infty}^{\infty} X(t) e^{-i2\pi f t} \, dt$$

$$= \lim_{T \to \infty} \int_{-T/2}^{T/2}$$

FACT: The F.T. can be computed very rapidly via "Fast Fourier Transform."

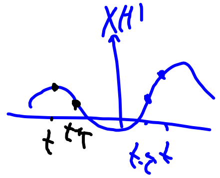
Review of ~~in~~ salient points of Fourier theory.

$$\text{Four. transf.}\left[\int_{-\infty}^{\infty} X(t)\, X(t+\tau)\, dt\right] = \left|\text{Four. Tran } X(t)\right|^2$$

minor point

$$\int_{-\infty}^{\infty} X(t)\, X(t-\tau)\, dt$$

same



PROBLEM (not unsolvable)

If $X(t)$ is stationary, random, zero mean,

for $\tau = 0$

$$\mathcal{E}\left[\int_{-\infty}^{\infty} X(t)\, X(t)\, dt\right] = \int_{-\infty}^{\infty} E[X(t)^2]\, dt$$

$$= \int_{-\infty}^{\infty} \sigma^2\, dt$$

$$= \infty$$

Terminology

$$\int_{-\infty}^{\infty} x(t)^2\, dt = \text{"energy"}$$

Fourier theory works on finite energy signals ("square-integrable", "L^2")

If $X(t)$ is random & stationary, its (expected) "energy" is $\sigma^2 \cdot \infty$.

Case i

energy $\lim_{T\to\infty} \int_{-T/2}^{T/2} X^2(t)\,dt$ $\qquad E[\text{this}] = \infty$

look at $\lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} X^2(t)\,dt$

$$E\left[\underline{\qquad\qquad\qquad} \right]$$

$$= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \sigma^2\,dt = \frac{\sigma^2 T}{T} = \sigma^2$$

THIS IS FINITE.

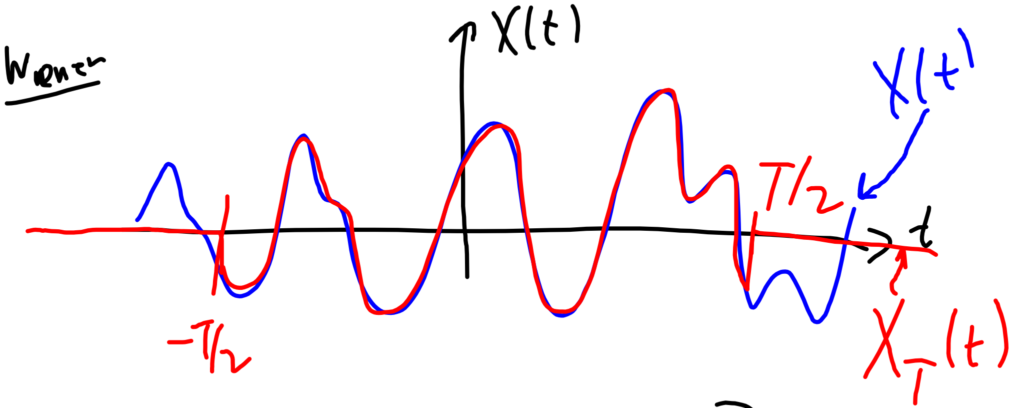
$\sigma^2 = E\left[X(t)^2\right]$ is called the power $\left(\frac{\text{energy}}{\text{time}}\right)$

So stationary random processes are not finite-energy, but <u>are</u> finite-power,

<u>Wiener</u> showed that you can use the Fourier tools to estimate $R_X(\tau)$ with a slight modification:

$$\text{Four. transf.}\left[\int_{-\infty}^{\infty} x(t)\, x(t+\tau)\, dt\right] = \left|\text{Four. Tran. } x(t)\right|^2$$

Wiener



$$\text{Four. Transf.}\left[\frac{1}{T}\int_{-\infty}^{\infty} x_T(t)\, x_T(t+\tau)\, dt\right] = \frac{1}{T}\left|\text{Four. Tran. }[x_T(t)]\right|^2$$

$$\text{F}\left[\int_{-\infty}^{\infty} x(t)\, x(t+\tau)\, dt\right] = \left|\text{Four}[x(t)]\right|^2$$

$$\text{Four Trans}\left[\frac{1}{T}\int_{-\infty}^{\infty} X_T(t)\,X_T(t+\tau)\,dt\right]$$

$$= \frac{1}{T}\left|\text{Four Trans}\left[X_T(t)\right]\right|^2$$

lim of these as $T\to\infty$

$$\lim_{T\to\infty} E\left[\text{left-hand side}\right] = FT\{R_X(\tau)\}$$

Right hand side : $\displaystyle\lim_{T\to\infty}\frac{1}{T}\left|FT\left[X_T(t)\right]\right|^2$

So the scheme for estimating $R_X(\tau)$ from a particular $X(t)$

$$R_X(\tau) \underset{\text{by}}{\overset{\text{is}}{\text{estimated}}} FT^{-1}\left\{ \frac{1}{T} \left| FT\left\{ X_T(t) \right\} \right|^2 \right\}$$

If you have 1024 samples of $X(t)$:

FFT of $X(t)$

absolute value

square

divide by 1024

inverse FFT

$\Longrightarrow R_X(\tau)$

(Refer to notes-jpegs.)

$R_x(\tau)$ autocorrelation function.

Four. Tran. $[R_x(\tau)]$ = "Power spectral density"

$= S(f)$

---

estimates $R_x(\tau) \simeq \frac{1}{T} \int_{-T/2}^{T/2} x(t)\, x(t+\tau)\, dt$ "correlogram"

"finite-time autocorrelation"

esitimate $S(f) \simeq \frac{1}{T} \left| FT[x(t)] \right|^2$

"Spectrogram"

$S(f)$ "PSD" is the mathematical

Four. Trans of the autocorrelation function $R_X(\tau)$.

BUT you don't compute ~~it~~ $S(\omega)$ in

the "natural" way:

1. $X(t)$
2. estimate $\frac{1}{T}\int x(t)\,x(t+\tau)\,dt$
3. Take F.T. of estimate.

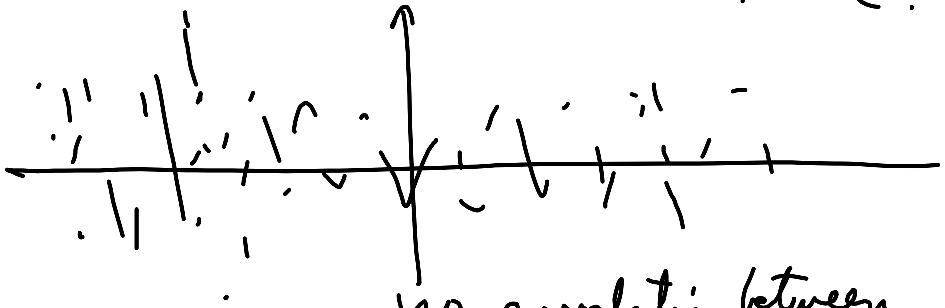
NO

Instead

1. $X(t)$
2. Take F.T. $X(t)$
3. Magnitude-square, divide by T.

Example.

WHITE NOISE.



no correlation between $X(t_1)$ & $X(t_2)$.

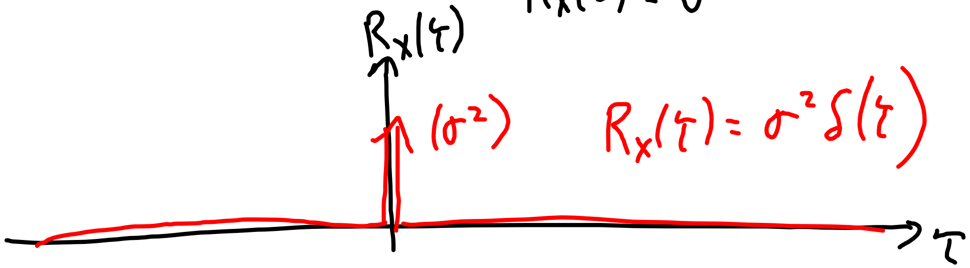
zero-mean, stationary.

$$R_X(\tau) = E\{X(t)X(t+\tau)\}$$

$$= E\{X(t)\}E\{X(t+\tau)\}$$

$$= 0 \quad \text{unless } \tau = 0$$

$$R_X(0) = \sigma^2$$

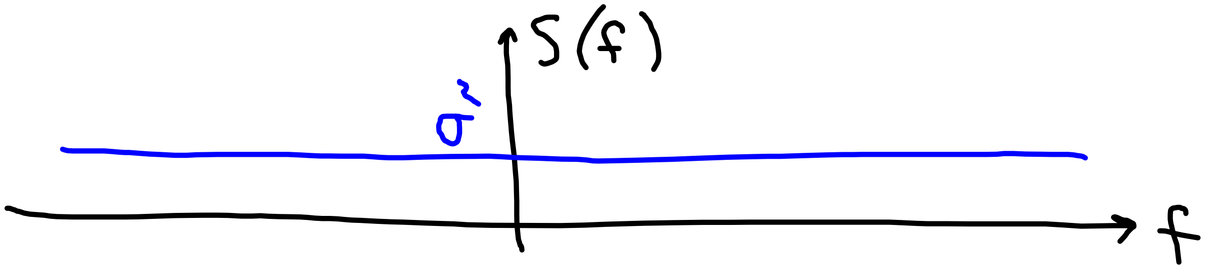


$$R_X(\tau) = \sigma^2 \delta(\tau)$$

What is PSD of white noise?

$$S(\omega) = \int_{-\infty}^{\infty} \underbrace{R_x(\tau)}_{\sigma^2 \delta(\tau)} e^{-i 2\pi f \tau} \, d\tau$$

$$= \sigma^2$$



Uniform "spectrum", called "white noise" by analogy with light.