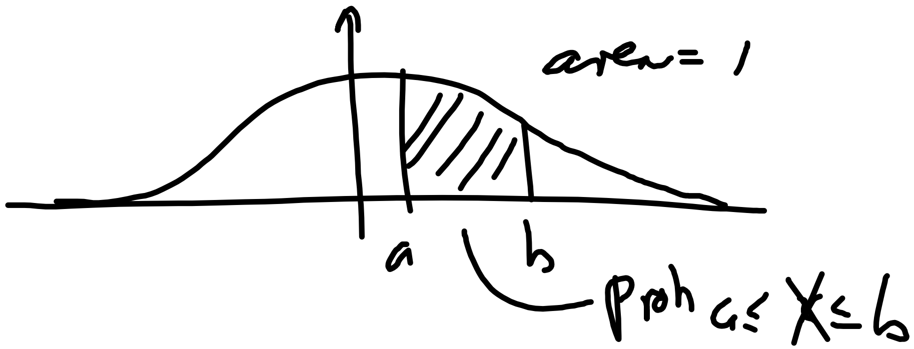
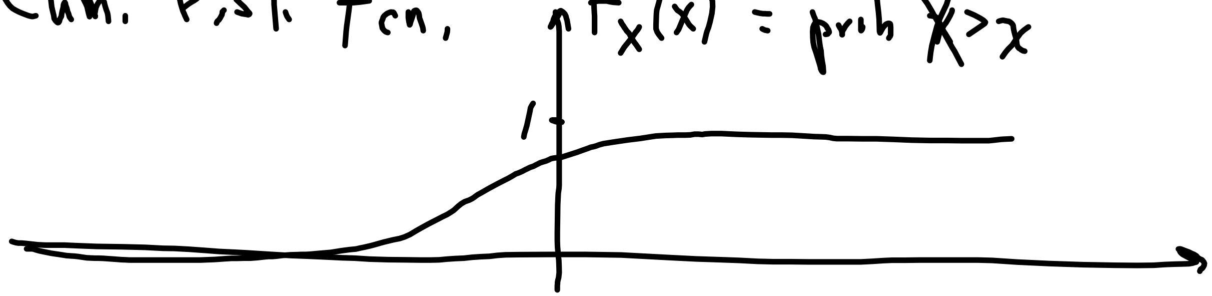
1

Prob. Dens. Fcn. $f_X(x)$



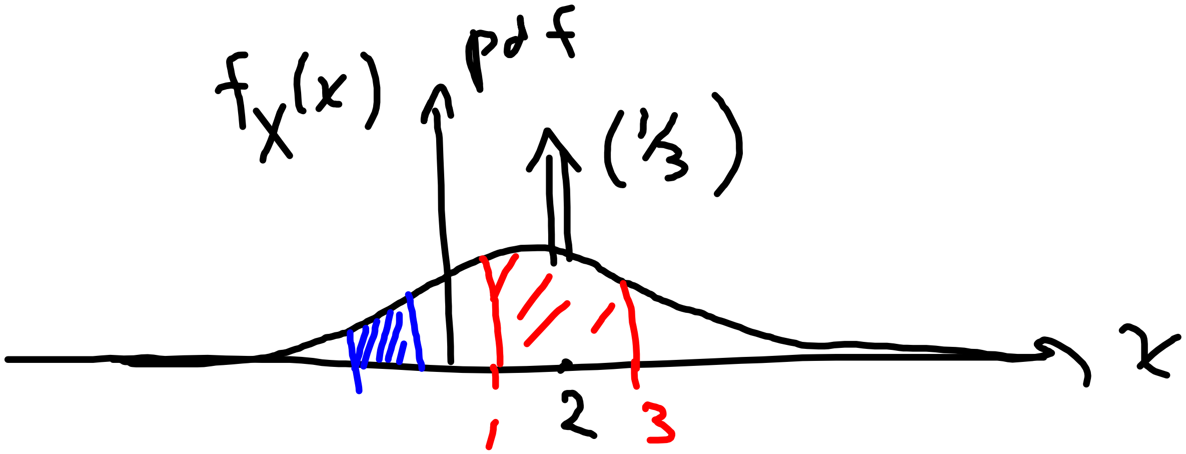
Cum. Dist. Fcn. $F_X(x) = \text{prob } X > x$



$$F_X(x) = \int_{-\infty}^{x} f_X(x)\,dx$$

$$f_X(x) = \frac{d F_X(x)}{dx}$$

What if there's a delta-fcn?



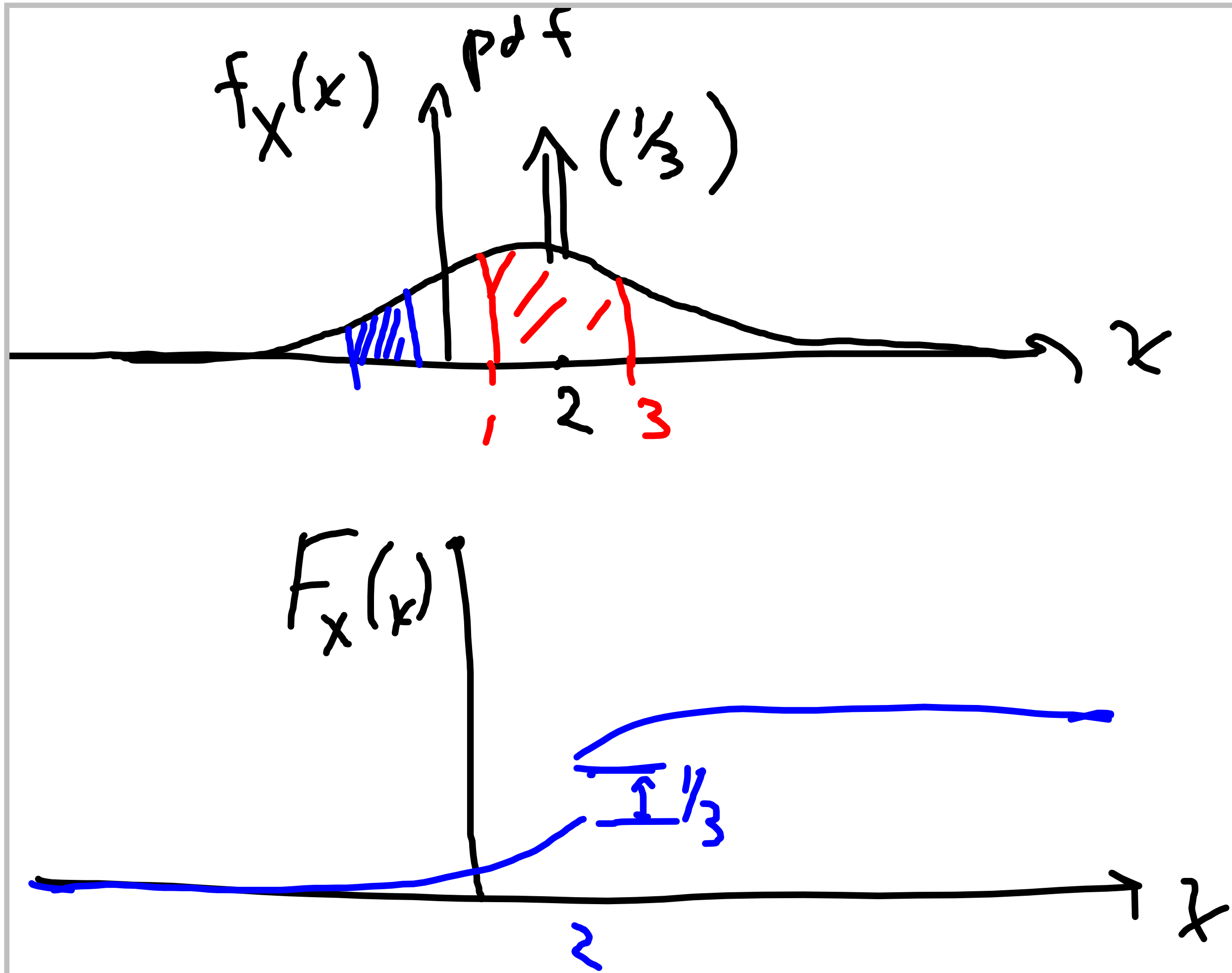
$\text{Prob}(1 \le X \le 3)$

$= \boxed{} + 1/3$

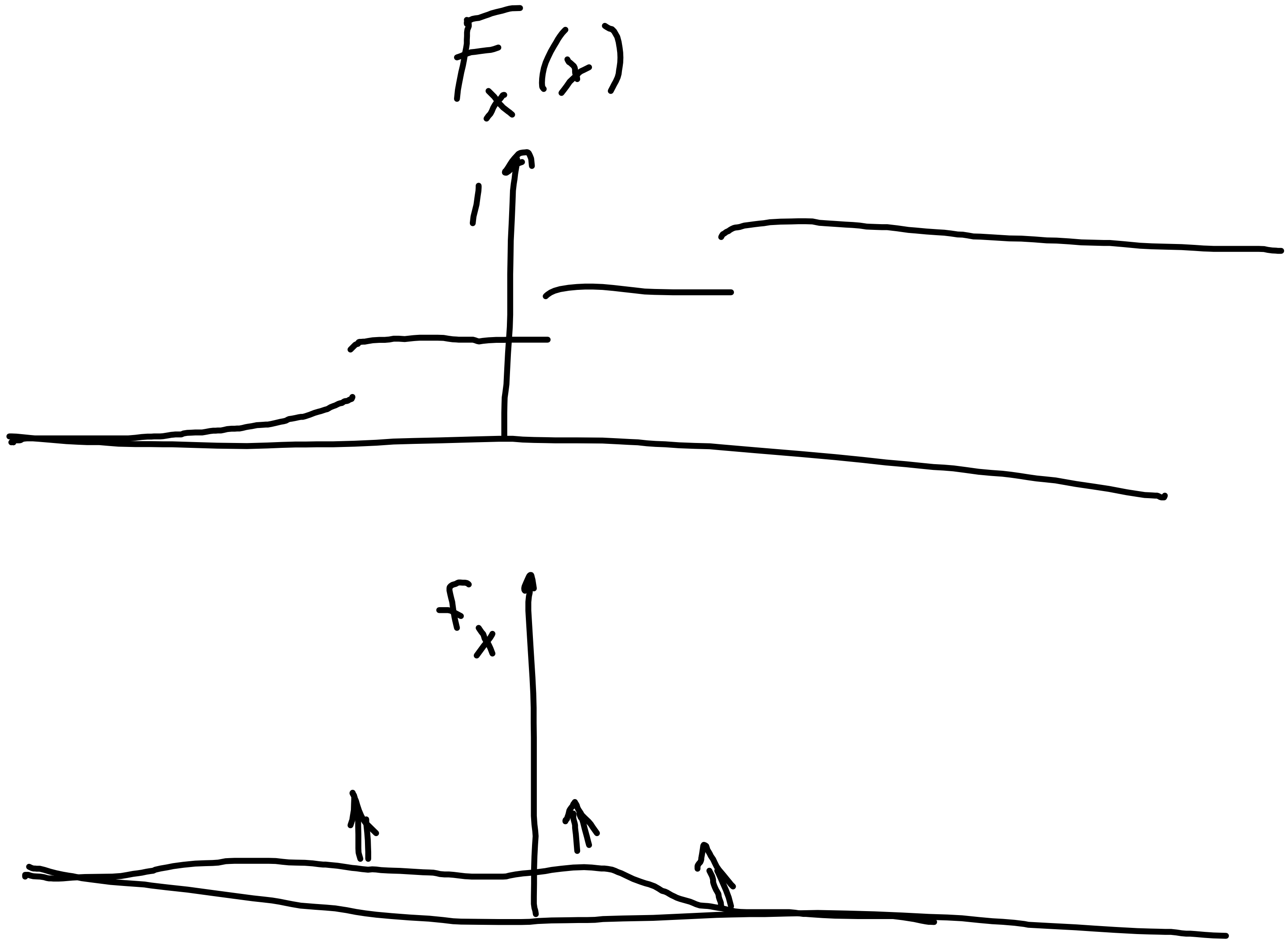
area

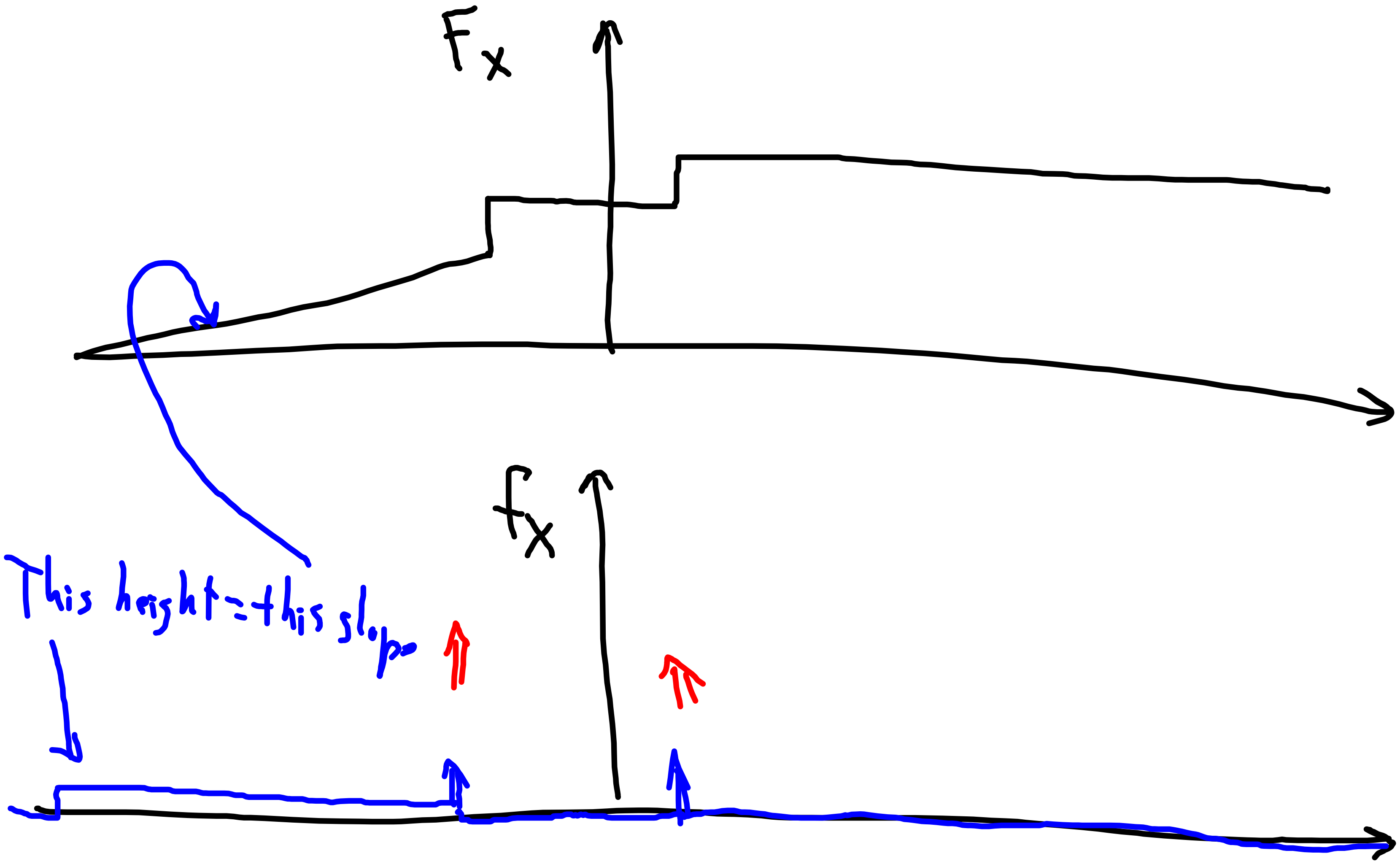
$\text{Prob}(X=2) = 1/3$

$\text{Prob}(X=3) = 0$



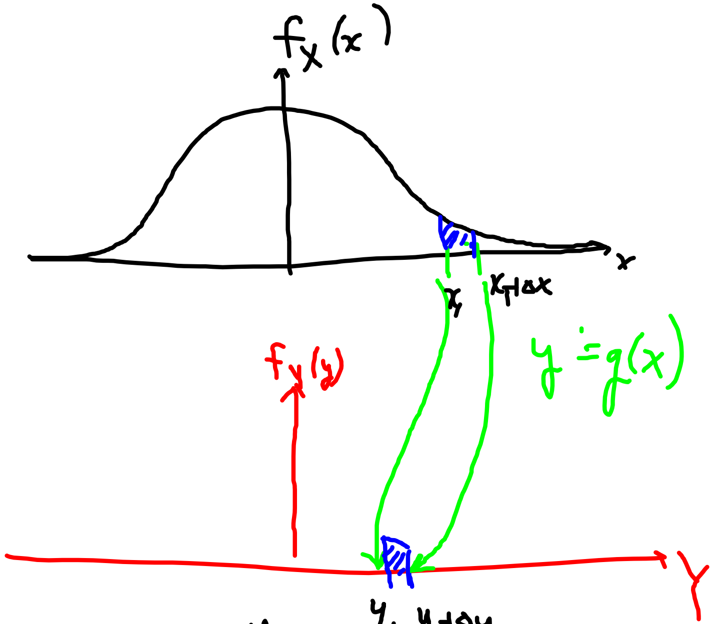
$f_X(x)$
pdf
(1/3)
x
1
2
3
$F_X(x)$
1/3
x
2

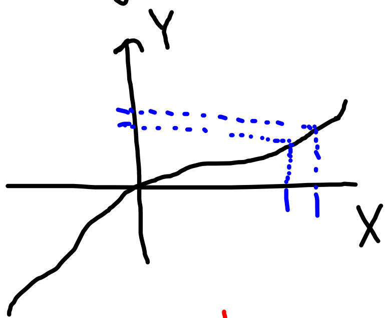

$F_x(x)$
1
f_x

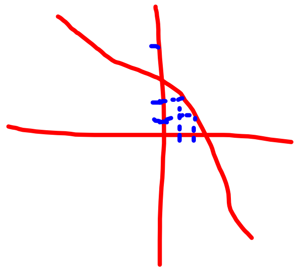

F_x
f_x
This height = this slope

Derived distributions. What is $f_Y(y)$ (in terms of f_X)?

$$Y = g(X)$$







You need $\displaystyle\int_{y_1}^{y_1+\Delta y} f_Y(y)\,dy = \int_{x_1}^{x_1+\Delta x} f_X(x)\,dx$

Added later: dy might be negative, so use bsolute value

$\Delta x, \Delta y$ small

$$f_Y(y)\,|\Delta y| = f_X(x)\,|\Delta x|$$

$$\boxed{f_Y(y) = \frac{f_X(x)}{|\Delta y/\Delta x|} \to \frac{f_X(x)}{|dy/dx|}}$$

Δx, Δy small

$$f_Y(y)\,\Delta y = f_X(x)\,\Delta x$$

$$\boxed{f_Y(y) = \frac{f_X(x)}{\left|\Delta y/\Delta x\right|} \longrightarrow \frac{f_X(x)}{\left|dy/dx\right|}}$$

The x's over here are the ones corresponding to the given y.

This is the slope of the g-curve

$$\frac{dy}{dx} = g'(x)$$

$\Delta x, \Delta y$ small

$$f_Y(y)\,|\Delta y| = f_X(x)\,|\Delta x|$$

$$\boxed{f_Y(y) = \frac{f_X(x)}{|\Delta y/\Delta x|} \longrightarrow \frac{f_X(x)}{|dy/dx|}}$$

$y = g(x)$

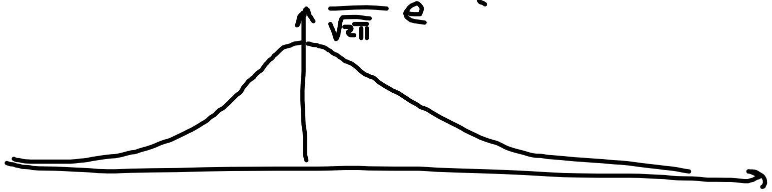
$x = g^{-1}(y)$

Better: $f_Y(y) = f_X(g^{-1}(y)) \Big/ |g'(g^{-1}(y))|$

$f_Y(y)\,dy = f_X(x)\,dx$ (mnemonic)

① $f_X = \text{Gaussian} = N(0,1)$

$\frac{1}{\sqrt{2\pi}} e^{-x^2/2}$



$Y = aX + b \qquad X = \frac{Y-b}{a}$

$$f_Y(y) = \frac{f_X(x)}{|dy/dx|} = \frac{f_X\left(\frac{y-b}{a}\right)}{|a|} = \frac{1}{\sqrt{2\pi}\,|a|} e^{-\frac{1}{2}\left(\frac{y-b}{a}\right)^2}$$

Elaborate on this

$Y = aX + b$

$\mathcal{E}(Y) = a\mathcal{E}(\cancel{X}) + \mathcal{E}(b) = b$

Always (Regardless of pdf)

$$\sigma_Y^2 = \mathcal{E}[(Y - \underbrace{b}_{\mathcal{E}(Y)})^2] = \mathcal{E}[(aX)^2] = a^2 \underbrace{\mathcal{E}[X^2]}_{1^2} = a^2$$

Scratch work:

$$\mathcal{E}[X^2] = \mathcal{E}(X)^2 + \sigma_X^2 \qquad X \text{ is } N(0,1)$$

$$\mathcal{E}(X)^2 = 0^2 \qquad \sigma_X^2 = 1^2$$

BOTTOM LINE: if X is $N(0,1)$

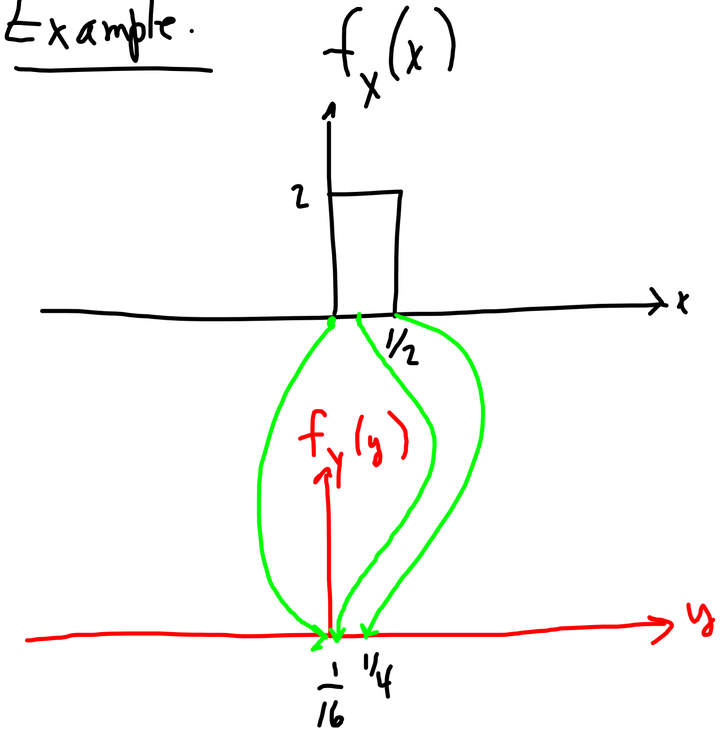
$$Y = aX + b \qquad \mathcal{E}(Y) = b \qquad \sigma_Y = |a|$$

$$f_Y(y) = \frac{1}{\sqrt{2\pi}\,|a|} e^{-\frac{1}{2}\left(\frac{y-b}{a}\right)^2} = \frac{1}{\sqrt{2\pi}\,|a|} e^{-\frac{(y-b)^2}{2a^2}}$$

is $N(b, a)$

Remember: if X is Gaussian then $aX+b$ is, also, Gaussian!

Example.



$$Y = X^2$$

$$f_Y|dy| = f_X|dx|$$

$$f_Y(y) = \frac{f_X(x)}{|dy/dx|} \leftarrow 2x$$

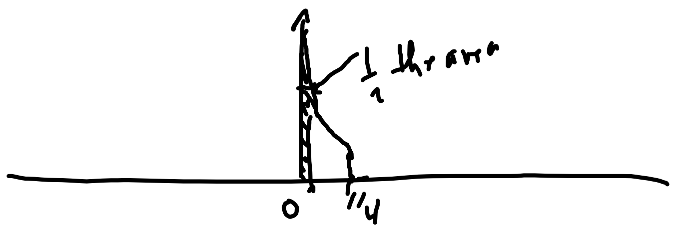
$$x = \sqrt{y}$$

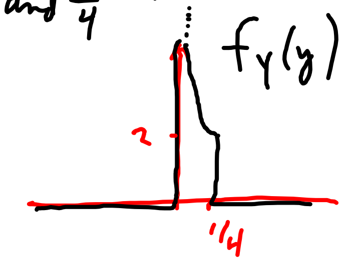
$$f_Y(y) = \frac{2}{2\sqrt{y}}$$

$$= \frac{1}{\sqrt{y}} \quad 0<y<\frac{1}{4}$$

$f_Y(y)$ will have $\frac{1}{2}$ it's area between 0 + $\frac{1}{16}$,

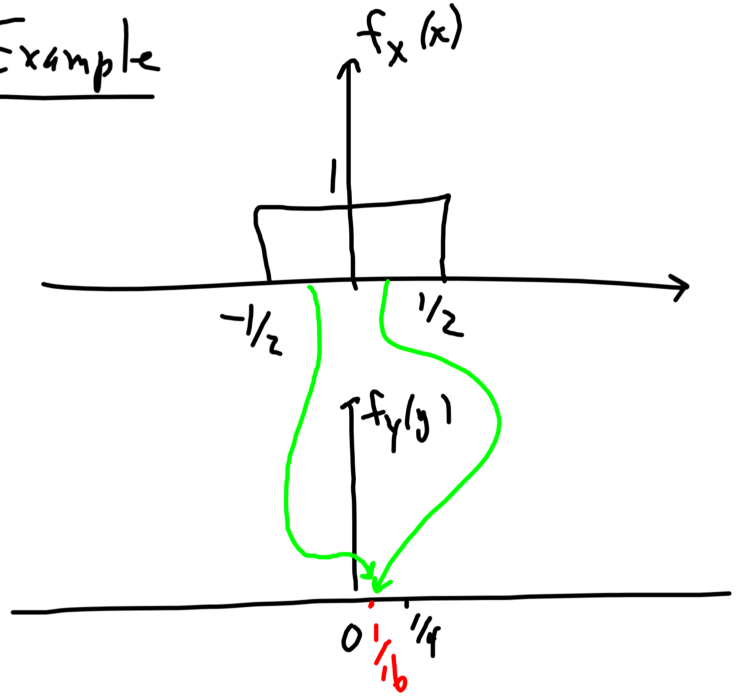
$\frac{1}{2}$ - — between $\frac{1}{16}$ and $\frac{1}{4}$.





Example

$$Y = X^2$$



$y = \frac{1}{16}$ comes from

$x = \frac{1}{4}$ and

$x = -1/4$.

We have to change $f_Y(y)|dy| = f_X(x)|dx|$

to $$f_Y(y)|dy| = f_X(x_1)|dx_1| + f_X(x_2)|dx_2|$$

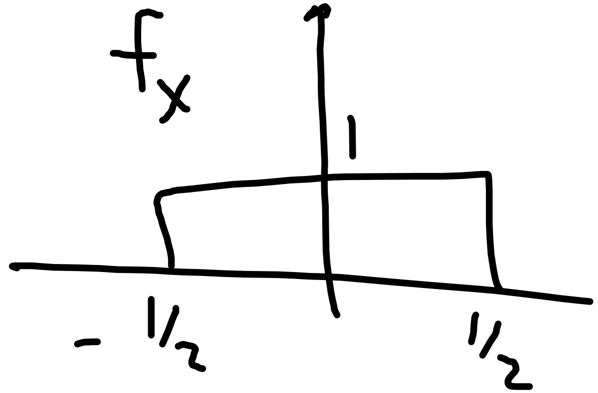
when $y = g(x_1)$
$= g(x_2)$

given y,

$$f_Y(y) = \sum_{\substack{\text{all } x \\ \text{that are} \\ \text{preimages} \\ \text{of } y}} \frac{f_X(x_i)}{\left|\frac{dy}{dx}\right|_{x_i}}$$

given y,

$$f_Y(y) = \sum_{\substack{\text{all } x \\ \text{that are} \\ \text{preimages} \\ \text{of}}} \frac{f_X(x_i)}{\left|\frac{dy}{dx}\right|_{x_i}}$$



$$Y = X^2$$

$$x_1 = \sqrt{y}$$

$$x_2 = -\sqrt{y}$$

$$= \frac{1}{\underset{\underset{\sqrt{y}}{\uparrow}}{|2x|}} + \frac{1}{\underset{\underset{-\sqrt{y}}{\uparrow}}{|2x|}}$$

$$= \frac{1}{\sqrt{y}} \quad 0 < y < \frac{1}{4}$$