$$\mathcal{E}[X+Y] = \mathcal{E}(X) + \mathcal{E}(Y)$$

(no exceptions)

$$\mathcal{E}[aX] = a\,\mathcal{E}[X]$$

$$Var[X+Y] = Var[X] + Var[Y] + 2\,\mathcal{E}\{[X-\mu_X][Y-\mu_Y]\}$$

<u>Proof</u>:

$$Var[X+Y] = \mathcal{E}\{[(X+Y) - \underbrace{\mathcal{E}(X+Y)}_{\mu_X+\mu_Y}]^2\}$$

$$= \mathcal{E}\{[X+Y-\mu_X-\mu_Y]^2\}$$

$$= \mathcal{E}\{[(X-\mu_X)+(Y-\mu_Y)]^2\}$$

$$= \mathcal{E}[(X-\mu_X)^2] + \mathcal{E}[(Y-\mu_Y)^2] + 2\,\mathcal{E}[(X-\mu_X)(Y-\mu_Y)]$$

$$= \sigma_X^2 + \sigma_Y^2 + 2\,\mathcal{E}[(X-\mu_X)(Y-\mu_Y)]$$

$$\mathrm{Var}[X+Y] = \mathrm{Var}[X] + \mathrm{Var}[Y] + 2\underbrace{\mathcal{E}\{[X-\mu_X][Y-\mu_Y]\}}_{\text{covariance}(X,Y)}$$

$$\sigma_{X+Y}^2 = \sigma_X^2 + \sigma_Y^2 + 2\underbrace{\mathrm{cov}(X,Y)}_{\rho\sigma_X\sigma_Y}$$

def. of ρ = "correlation coefficient."

If X + Y vary about their mean independently, expect $\mathcal{E}\{(X-\mu_x)(Y-\mu_y)\} \simeq 0$

If X + Y are correlated, $X \simeq 2Y + 7$,

expect $\mathcal{E}[(X-\mu_x)(Y-\mu_y)] > 0$

If X + Y are "anticorrelated", $X = -2Y + 7$

expect $\mathcal{E}\{(X-\mu_x)(Y-\mu_y)\} < 0$

Fact. (p. 175)

If X & Y are independent,

$$\rho = 0$$

$$\cancel{\text{also } E[XY]}$$

If $X = aY + b$, $\quad \rho = 1$ if $a > 0$

$\rho = -1$ if $a < 0$

$$\operatorname{cov} \cancel{\operatorname{Cor}}(X,Y) \stackrel{\text{def}}{=} \mathcal{E}\{(X-\mu_X)(Y-\mu_Y)\}$$

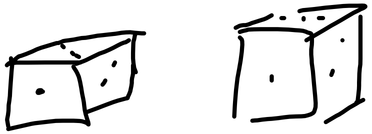
Fact.

$$\mathcal{E}\{XY\} = \operatorname{cov}(X,Y) + \mu_X \mu_Y$$

is an extension of

$$\mathcal{E}(X^2) = \mathcal{E}[(X-\mu_X)^2] + \mu_X^2$$

$$\text{"correlation}(X,Y)\text{"} = \mathcal{E}(XY)$$



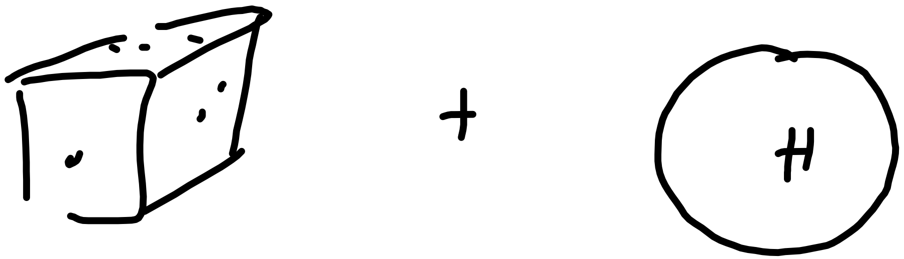
$$P(1) = P(2) \ldots = P(6) = 1/6$$

$X \qquad Y$

$$P(X+Y=9) = P(X=6 \,\&\, Y=3) = \tfrac{1}{6}\cdot\tfrac{1}{6}$$
$$+\,P(X=5 \,\&\, Y=4) = \tfrac{1}{6}\cdot\tfrac{1}{6}$$
$$+\,P(X=4 \,\&\, Y=5) = \tfrac{1}{6}\cdot\tfrac{1}{6}$$
$$+\,P(X=3 \,\&\, Y=6) = \tfrac{1}{6}\cdot\tfrac{1}{6}$$

$$+\,P(X=2 \,\&\, Y=7) = \tfrac{1}{6}\cdot 0$$
$$\vdots$$

$$P(X+Y=9) = \sum_{x} P(X=x \,\&\, Y=9-x)$$



1 or 0

$$\text{Prob}(X+Y=4) = P(X=3, Y=1) \quad \frac{1}{6}\cdot\frac{1}{2}$$

$$+ P(X=4, Y=0) \quad \frac{1}{6}\cdot\frac{1}{2}$$

$$+ \text{'impossible'}$$

$$= \sum_{x} P(X=x,\ Y=4-x) \quad ..$$

Given w

$$P(X+Y=w) = \sum_{x} P(X=x \;\&\; Y=w-x)$$

if independent $$= \sum_{x} P(X=x)\, P(Y=w-x)$$

if not $$= \sum_{x} P(X=x)\, P(Y=w-x \mid X=x)$$

$$f_{X+Y}(w) = \int_{-\infty}^{\infty} f_X(x)\, f_Y(w-x)\, dx$$

density formula for the sum of independent random variables.

$$f_{X+Y}(w) = \int_{x=-\infty}^{\infty} f_X(x)\, f_Y(w-x)\, dx$$

$$= \int_{y=-\infty}^{\infty} f_Y(y)\, f_X(w-y)\, dy$$

called the "convolution" of f_X & f_Y.

$$f_{X+Y+Z}(\omega) = \mathrm{conv}\left(f_Z,\ f_{X+Y}\right)$$

$$f_{X+Y} = \mathrm{conv}(f_X, f_Y)$$

<u>Fact.</u> If X is $N(\mu_X, \sigma_X)$ & Y is $N(\mu_Y, \sigma_Y)$, and X & Y independent then $X+Y$ is Gaussian

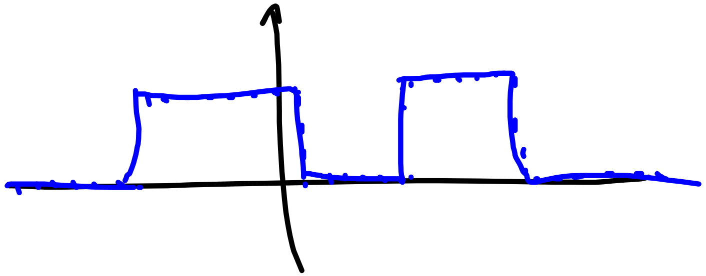
Central Limit Theorem.

$$X_1, X_2, X_3, \dots X_n$$

independent, some density function "identical, independently distributed", "i.i.d.", (not necessarily Gaussian), then $\sum_1^n X_j \approx$ Gaussian.

MATLAB Experiment.

$f_x =$



A = [zeros(1,10) ones(1,5) zeros(1,5), ones(1,5),
zeros(1,10)];

plot A;

B = A; C = conv(A,B); plot C

B = A; B = conv(A,B); plot B.

Central Limit Theorem.

$$X_1, X_2, X_3, \ldots X_n$$

independent, same density function "identical, independently distributed" "i.i.d.", (not necessarily Gaussian), then $\sum_1^n X_j \approx$ Gaussian.

mean $= n\mu_X$ ~~st. dev.~~ $=$

var $= n\sigma_X^2$

st dev $= \sqrt{n}\,\sigma_X$

average $= \dfrac{\sum_1^n X_j}{n}$ mean $= \mu_X$

var $= n\left(\dfrac{\sigma_X}{n}\right)^2 = \dfrac{1}{n}\sigma_X^2$

st dev $= \dfrac{1}{\sqrt{n}}\sigma_X$