# Linear Estimator.

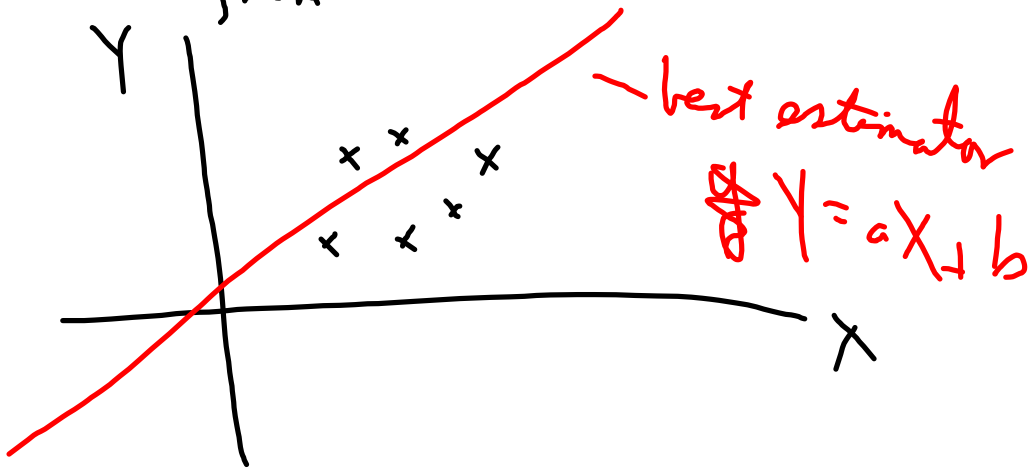
Have 2 random processes:

$$X(n), \quad Y(n).$$

Suspect they are linearly related

$$Y = aX + b, \text{ except for randomness.}$$

Scatter diagram



$Y = aX + b$ What are a, b?

error $Y - [aX+b]$

minimize $\mathcal{E}[\text{error}] = \mathcal{E}\left[Y - (aX+b)\right]$

no: never minimize an unsigned number.

We will minimize $\mathcal{E}[\text{error}^2]$.

Least Mean Square LMS

LMS procedure for finding best linear estimator.

minimize $\mathcal{E}\left[(Y-aX-b)^2\right]$

$$\mathcal{E}\left[(\quad)^2\right] = \mathcal{E}\left[Y^2 + a^2X^2 + b^2 - 2aYX - 2bY + 2abX\right]$$

$$= \mathcal{E}[Y^2] + a^2\mathcal{E}[X^2] + b^2 - 2a\,\mathcal{E}[YX] - 2b\,\mathcal{E}[Y] + 2ab\,\mathcal{E}[X]$$

at minimum:

$$\frac{\partial}{\partial a}\mathcal{E}\left[(Y-aX-b)^2\right] = 0 = 2a\,\mathcal{E}[X^2] - 2\,\mathcal{E}[YX] + 2b\,\mathcal{E}[X]$$

$$\mathcal{E}[X^2]\,a + \mathcal{E}[X]\,b = \mathcal{E}[YX]$$

$$\frac{\partial}{\partial b}\mathcal{E}\left[(Y-aX-b)^2\right] = 0 = 2b + 2a\,\mathcal{E}[X] - 2\,\mathcal{E}[Y]$$

$$\mathcal{E}[X]\,a + b = \mathcal{E}[Y]$$

$$\begin{bmatrix} \mathcal{E}[X^2] & \mathcal{E}[X] \\ \mathcal{E}[X] & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \mathcal{E}[YX] \\ \mathcal{E}[Y] \end{bmatrix}$$

$$a = \frac{\mathcal{E}[YX] - \mathcal{E}[X]\mathcal{E}[Y]}{\mathcal{E}[X^2] - \mathcal{E}[X]^2} = \frac{\mathcal{E}[YX] - \mu_x \mu_y}{\sigma_x^2}$$

$$= \frac{\mathcal{E}[(Y-\mu_y)(X-\mu_x)]}{\sigma_x^2}$$

$$= \frac{\mathrm{cov}[Y,X]}{\sigma_x^2}$$

$$= \frac{\rho\, \sigma_x \sigma_y}{\sigma_x^2}$$

$$a = \rho \frac{\sigma_Y}{\sigma_x}$$

$$\begin{bmatrix} \mathcal{E}[X^2] & \mathcal{E}[X] \\ \mathcal{E}[X] & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \mathcal{E}[YX] \\ \mathcal{E}[Y] \end{bmatrix}$$

$$b = \frac{\mathcal{E}[X^2]\,\mathcal{E}[Y] - \mathcal{E}[X]\,\mathcal{E}[YX]}{\sigma_X^2}$$

$$= \frac{[\mu_X^2 + \sigma_X^2]\mu_Y - \mu_X[\rho\,\sigma_Y\sigma_X + \mu_X\mu_Y]}{\sigma_X^2}$$

$$b = \mu_Y - \mu_X\,\rho\,\sigma_Y/\sigma_X$$

Finally: best ~~ell~~ mean-square linear estimator is

$$\boxed{Y = \rho \frac{\sigma_Y}{\sigma_X} X + \mu_Y - \rho \frac{\sigma_Y}{\sigma_X} \mu_X}$$

Unbiased? (left-hand mean = right-hand mean?)

YES: $\mu_Y = \mu_Y$.

Quadratic estimator:

Random processes $X(n), Y(n)$:

Estimator $Y = aX^2 + b\cancel{X} + c$

same procedure: $\frac{\partial}{\partial a,b,c} E\left[(Y - aX^2 - bX - c)^2\right] = 0$

Estimator $Y = a \sin X + b \cos X$

same procedure.

Estimator $Y = a \sin \omega X + b \cos \omega X$

(find $a, b, \omega$)

very difficult: equation are nonlinear in $\omega$.

Estimator: $Y = a e^{bX}$

nonlinear; but $\ln Y = \underbrace{\ln a}_{a'} + bX$

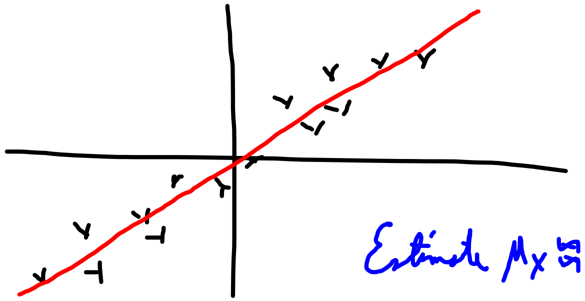
Estimator $Y = a e^{-bX} + d e^{-cX}$

Use special methods (Prony)

Back to $Y = aX + b$.

have data $X(1) \ldots X(N)$

$Y(1) \ldots Y(N)$



Estimate $\mu_X \simeq \frac{\sum X(j)}{N}$ $\mu_Y \simeq \frac{\sum Y(j)}{N}$

$$\sigma_X^2 \simeq \frac{1}{N} \sum X(j)^2 - \mu_X^2$$

$$\sigma_Y^2 = \frac{1}{N} \sum Y(j)^2 - \mu_Y^2$$

$$\rho = \frac{\sum X(j) Y(j) - \mu_X \mu_Y}{\sigma_X \sigma_Y}$$

# Wiener filtering

measurements of $x(n)$

contains noise $v(n)$

want to extract $d(n)$

Estimator $d(n) \simeq \underline{w(-1)}x(n+1) + \underline{w(0)}x(n) + \underline{w(1)}x(n-1)$
$+ \underline{w(2)}x(n-2)$

Task: what values should the w's take to minimize the expected value of the squared error in the estimate?

~~Extra Cg~~

First application: extract a signal from noise.

$d(n)$ desired signal.

$$x(n) = d(n) + v(n)$$

Estimate $d(n)$ from the measurements:

$$d(n) = w(0)\,x(n) + w(1)\,x(n-1)$$

Procedure:

1. Write formula for error (LHS - RHS)
2. Square " " "
3. Take $\mathcal{E}$ [squared error] = ..... $w(0)$ .... $w(1)$.....
4. Minimize $\frac{\partial}{\partial w(0)} = 0 \quad \frac{\partial}{\partial w(1)} = 0$
5. Solve for $w(0)$, $w(1)$.