Kalman Filter

To estimate $X(1)$

Ingredients:

1. Measurement of $X(1)$:

$$Y(1) = X(1) + V_1^{meas}$$

error, independent, mean zero, variance σ^2_{meas}

2. Model for $X(1)$ in terms of $X(0)$:

"Equation of motion"

"state equation"

(model)

$$X(1) = \ldots X(0) \ldots + w_1^{model}$$

zero mean

variance σ^2_{mod}

3. Unbiased estimate of

$X(0)$:

$$X(0) = \hat{X}(0) + e_0^{error}$$

zero mean: $E(e_0^{err}) = 0$

Variance

$$E({e_0^{err}}^2) = \sigma_0^2$$

From $\hat{X}(0)$ and the model you construct a predictor of $X(1)$:

Predictor: predict

$X(1)$ from $X(0)$.

If the model is

$$X(1) = A_0 X(0) + w_1^{mod}$$

and you have an estimate

$$\hat{X}(0) = X(0) + e_0^{err},$$

predictor for $X(1)$ is

$$A_0 \hat{X}(0).$$

GOAL:

$$\hat{X}(1) = K Y(1) + L A_0 \hat{X}(0)$$

choose K and L so that

1. $\hat{X}(1)$ is unbiased
2. mean square error of $\hat{X}(1)$ is minimal

$$\hat{X}(1) = K Y(1) + L A_0 \hat{X}(0)$$

$$= A_0 \hat{X}(0) + K\left[Y(1) - A_0 \hat{X}(0)\right]$$

since $L = 1 - K$

We will see that the optimal "Kalman gain" K is

$$K = \frac{\sigma_{model}^2 + A_0 \sigma_0^2}{\sigma_{model}^2 + A_0 \sigma_0^2 + \sigma_{meas}^2}$$

1. We have seen how to find each of these parameters for the old final problems.
2. $0 \le K \le 1$, so $0 \le L \le 1$.

   $\hat{X}(1)$ lies between the measurement $Y(1)$ and the predictor $A_0 \hat{X}(0)$

$$\hat{X}(1) = K Y(1) + \underset{\uparrow \atop 1-K}{L} A_0 \hat{X}(0)$$

We will prove

$$K = \frac{\sigma_{mod}^2 + A_0 \sigma_0^2}{\sigma_{mod}^2 + A_0 \sigma_0^2 + \sigma_{meas}^2}$$

3. If the measurement is perfect,

$\sigma_{meas} = 0 \Rightarrow K = 1 \Rightarrow$

$\hat{X}(1) = Y(1)$ (of course)

4. If the measurement is horrible,

$\sigma_{meas} = \infty \Rightarrow K = 0 \Rightarrow$

$\hat{X}(1) = A_0 \hat{X}(0)$ (of course)

5. The mean square error in $\hat{X}(1)$ is easily computed. So you have what you need to estimate $X(2), \ldots$

Derivation of $K_{optimal}$.

$$\hat{X}(1) = A_0 \hat{X}(0) + K[Y(1) - A_0 \hat{X}(0)]$$

$$e_1^{err} = \hat{X}(1) - X(1) \qquad \mathcal{E}[e_1^{err}] = 0$$

$$\mathcal{E}[e_1^{err\,2}] =$$

$$\mathcal{E}\left[\left(A_0 \hat{X}(0) - X(1) + K[Y(1) - A_0 \hat{X}(0)]\right)^2\right]$$

$$= \mathcal{E}[(\alpha + K\beta)^2]$$

$$= \mathcal{E}[\alpha^2 + 2K\alpha\beta + K^2\beta^2]$$

$$= \mathcal{E}[\alpha^2] + 2K\mathcal{E}[\alpha\beta] + K^2\mathcal{E}[\beta^2]$$

$$= A + 2KB + CK^2$$

At the minimum,

$$\frac{d}{dK}\mathcal{E}[e_2^{err\,2}] = 0 = 2B + 2CK$$

$$\Rightarrow K_{min} = -B/C$$

$$K = -B / C$$

$$B = \mathcal{E}[\alpha \beta] \quad C = \mathcal{E}[\beta^2]$$

$$\alpha = A_0 \hat{X}(0) - X(1)$$

$$= A_0 [X(0) + e_0] - [A_0 X(0) + w_1^{mod}]$$

$$= A_0 e_0 - w_1^{mod}$$

$$\beta = Y(1) - A_0 \hat{X}(0)$$

$$= X(1) + v_1^{meas} - A_0 [X(0) + e_0]$$

$$= A_0 X(0) + w_1^{mod} + v_1^{meas} - A_0 X(0) - A_0 e_0$$

$$= w_1^{mod} + v_1^{meas} - A_0 e_0$$

$$\alpha\beta = \left(A_0 e_0 - w_1^{mod}\right)\left(w_1^{mod} + v_1^{meas} - A_0 e_0\right)$$

Since all the errors are uncorrelated,

$$\mathcal{E}[\alpha\beta] = -A_0^2\, \mathcal{E}[e_0^2] - \mathcal{E}[w_1^2] + 0$$

$$= -A_0^2 \sigma_0^2 - \sigma_{mod}^2 = B$$

$$\mathcal{E}[\beta^2] = \mathcal{E}\left[\left(w_1^{mod} + v_1^{meas} - A_0 e_0\right)^2\right]$$

$$= \sigma_{mod}^2 + \sigma_{meas}^2 + A_0^2 \sigma_0^2 = C$$

$$K_{Kalman} = -\frac{B}{C} = \frac{\sigma_{mod}^2 + A_0^2 \sigma_0^2}{\sigma_{mod}^2 + A_0^2 \sigma_0^2 + \sigma_{meas}^2}$$

Recall $\hat{X}(1) = X(1) + e_1^{err}$

$$\mathcal{E}\left[e_1^{err^2}\right] = A + 2KB + K^2C$$

$$K_{min} = -B/C$$

so

$$\mathcal{E}\left[e_1^{err^2}\right] = \sigma_1^2 = A + 2\left(\frac{-B}{C}\right)B + \left(\frac{B}{C}\right)^2 C$$

$$= A - B^2/C$$

$$A = \mathcal{E}[\alpha^2] = \mathcal{E}\left\{\left(A_0 e_0 - w_1^{mod}\right)^2\right\}$$

$$= A_0^2 e_0^2 + \sigma_{mod}^2 = -B$$

$$\sigma_1^2 = -B - \frac{B^2}{C} = -B\left[\frac{C+B}{C}\right]$$

$$= -\frac{B}{C}[C+B]$$

$$\boxed{\sigma_1^2 = K\sigma_{meas}^2}$$

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Wiener Filter

STATIONARY Random zero-mean Processes

$$d(1), d(2), d(3), \ldots$$

$$x(1), x(2), x(3), \ldots$$

GOAL: to estimate the d's from the x's.

Ingredients:

1. You know the autocorrelations of the d's:

$$r_d(k) = \mathcal{E}\left[d(n)\, d(n-k)\right]$$

(For example, the d's come out of an ARMA process.)

2. You know how the d's are related to the x's

For example,

$$x(n) = d(n) + v(n)$$

zero-mean noise, σ_v^2, independent of all the other noises.

3. So you can compute the cross-correlations between x's and d's:

$$r_{dx}(k) = \mathcal{E}\left[d(n)\,\underset{\substack{\uparrow \\ d(n-k)+v(n-k)}}{x(n-k)}\right]$$

$$= \mathcal{E}\left[d(n)\,d(n-k)\right]$$

$$= r_d(k)$$

4. And you can compute the autocorrelations of the x's:

$$r_x(k) = \mathcal{E}\left[x(n)\,x(n-k)\right]$$

$$= \mathcal{E}\left[\left(d(n)+v(n)\right)\left(d(n-k)+v(n-k)\right)\right]$$

$$= \mathcal{E}\left[d(n)\,d(n-k)\right] + 0 + 0$$

$$+ \mathcal{E}\left[v(n)\,v(n-k)\right]$$

$$= r_d(k) + \sigma_v^2\,\delta_{k,0}$$

With all these correlations at hand, what is the best estimate of $d(n)$ in terms of a collection of $\{x(n)\}$?

#1 Prediction:

$$\hat{d}(n) = \alpha x(n-1) + \beta x(n-2)$$

#2 Smoothing:

$$\hat{d}(n) = \alpha x(n-1) + \beta x(n) + \gamma x(n+1)$$

#3 Filtering:

$$\hat{d}(n) = \alpha x(n) + \beta x(n-1) + \gamma x(n-2) \quad \cdots$$

Filtering Example.

$$\hat{d}(n) = \alpha x(n) + \beta x(n-1)$$

Since everything is zero-mean, the filter is unbiased.

Goal: find values of α and β which minimize the Mean Square Error

$$MSE = \mathcal{E}\left[(\hat{d}(n) - d(n))^2 \right]$$

$$= \mathcal{E}\left[(\alpha x(n) + \beta x(n-1) - d(n))^2 \right]$$

$$= \alpha^2 \mathcal{E}\left[x(n)^2\right] + \beta^2 \mathcal{E}\left[x(n-1)^2\right] + \mathcal{E}\left[d(n)^2\right]$$

$$+ 2\alpha\beta\, \mathcal{E}\left[x(n)x(n-1)\right] - 2\alpha \mathcal{E}\left[x(n)d(n)\right]$$

$$- 2\beta\, \mathcal{E}\left[x(n-1)d(n)\right]$$

$$= \alpha^2 r_x(0) + \beta^2 r_x(0) + r_d(0)$$

$$+ 2\alpha\beta\, r_x(1) - 2\alpha\, r_{dx}(0)$$

$$- 2\beta\, r_{dx}(1)$$

$$\alpha^2 r_x(0) + \beta^2 r_x(0) + r_d(0)$$

$$+ 2\alpha\beta\, r_x(1) - 2\alpha\, r_{dx}(0)$$

$$- 2\beta\, r_{dx}(1) = f(\alpha, \beta)$$

At the minimum,

$$\frac{\partial f}{\partial \alpha} = 0 \qquad \frac{\partial f}{\partial \beta} = 0$$

$$2\alpha\, r_x(0) + 2\beta\, r_x(1) - 2 r_{dx}(0) = 0$$

$$2\beta\, r_x(0) + 2\alpha\, r_x(1) - 2 r_{dx}(1) = 0$$

$$\begin{bmatrix} r_x(0) & r_x(1) \\ r_x(1) & r_x(0) \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} r_{dx}(0) \\ r_{dx}(1) \end{bmatrix}$$

We will see that the general WIENER-HOPF equations always take the form

$$\begin{bmatrix} \mathbb{R}_x \end{bmatrix} \begin{bmatrix} \text{coeffs} \end{bmatrix} = \begin{bmatrix} r_{dx} \end{bmatrix}$$