Students: When I taped this lecture, the SmartBoard writer was misaligned and I had great difficulty creating a legible set of notes. (You will become aware of this as you watch the lecture.) So this is a revised set of notes; it contains every equation that you will see on the screen, but it is not an exact copy.

X random variable

$$X = \hat{X} + e$$

estimate + error

Assume $E[e] = 0$

$\Longleftrightarrow$ $\hat{X}$ is <u>unbiased</u>

$\Longleftrightarrow$ $E[X] = E(\hat{X})$

$$E[e^2] = \sigma_{error}^2$$

$$25.6 \pm .2$$

$\uparrow$

Root Mean Square

$$\sqrt{E[e^2]}$$

$$E(e^2) = .2^2 = .04$$

Kalman Filter

To estimate $X(1)$

Ingredients:

1. <u>Measurement</u> of $X(1)$:

$$Y(1) = X(1) + v_1^{meas}$$

error,
independent,
mean zero,
variance σ^2_{meas}

2. <u>Model</u> for $X(1)$ in terms of $X(0)$:

"Equation of motion"
"state equation"

(model)

$$X(1) = \cdots X(0) \cdots + w_1^{model}$$

zero mean

variance σ_{mod}^2

3. Unbiased estimate of $X(0)$:

$$X(0) = \hat{X}(0) + e_0^{error}$$

zero mean: $E(e_0^{err}) = 0$

Variance
$$E({e_0^{err}}^2) = \sigma_0^2$$

From $\hat{X}(0)$ and the model you construct a predictor of $X(1)$:

Predictor: predict

$X(1)$ from $X(0)$.

If the model is

$$X(1) = A_0 X(0) + w_1^{mod}$$

and you have an estimate

$$\hat{X}(0) = X(0) + e_0^{err},$$

predictor for $X(1)$ is

$$A_0 \hat{X}(0).$$

GOAL:

$$\hat{X}(1) = K Y(1) + L A_0 \hat{X}(0)$$

choose K and L so that

1. $\hat{X}(1)$ is unbiased
2. mean square error of $\hat{X}(1)$ is minimal

EEL 6545 Final Exam Dec. 1, 2004 11:50 - 1:50. 2 hours. Open books and notes.

3. Two jewelers make measurements of the weight of a gold earring. The first jeweler reports a value of 50.0 grams, with an estimated *root* mean squared error of 0.6 gram; in other words, the estimated *squared* error is 0.36 . The second measurement results in a value of 50.5 grams, with an estimated *root* mean squared error of .8 grams. What is the best (Kalman) estimate of the actual weight of the earring, and what is its mean squared error? Is your answer close to what you would expect? Discuss.

4. Suppose the measurements in Problem 3 were done in the reverse order - the 50.5±0.8 was done first, then the 50.0±0.6 was done second. Work out the Kalman estimate for the weight of the earring. Is your answer different from #3? Discuss.

Measurement #1:

weight $= 50.0 \pm 0.6 \leftarrow X(0)$

Measurement #2:

weight $= 50.5 \pm 0.8 \leftarrow X(1)$

<u>model</u> $X(1) = \cdots X(0) \cdots$

$$X(1) = \underbrace{X(0)}_{A_0 = 1} + \underbrace{0}_{w_1^{meas} = 0}$$

<u>measurement</u>

$Y(1) = 50.5 \quad \sigma_1^{meas} = 0.8^2 = .64$

<u>estimate</u> of $\hat{X}(0) = 50.0$

$\sigma_0^2 = 0.6^2 = .36$

KALMAN

$$\hat{X}(1) = K\,Y(1) + L\,\underbrace{A_0}_{1}\,\hat{X}(0)$$

3. Suppose that in one day a radioactive substance decays to half its weight (its half-life is one day). Yesterday at noon a sample was measured to weigh 100±5 grams. You may assume that 100 grams was an unbiased estimate of its true weight and that 5 grams was the square root of this estimate's expected squared error.
Today at noon, its weight was measured to be 49±3 grams. Again, you may assume that this specifies an unbiased estimate and its root-mean-square error.
A note has just been discovered stating that vandals had altered the sample by 1 gram just before today's weighing; we don't know whether the 1g was added or subtracted, and either possibility seems equally likely.
What is the Kalman estimate of the sample's true weight at noon today? What is the expected value of the squared error in this estimate?

4. Working backward, use the data of problem 3 to calculate the Kalman estimate of the *original* weight of the sample, at noon yesterday.

$100 \pm 5 \qquad \hat{X}(0) = 100 \quad \sigma_0^2 = 25$

$49 \pm 3 \qquad Y(1) = 49 \quad \sigma_{meas}^2 = 9$

$X(1) = \frac{1}{2} X(0) \pm 1$

$X(1) = \frac{1}{2} X(0) + w_1^{mod}$

$\pm 1 \Rightarrow E[w_1] = 0$

$\sigma_{mod}^2 = \frac{1}{2}(1)^2 + \frac{1}{2}(-1)^2 = 1$

$\hat{X}(1) = K\,Y(1) + L\,A_0\,\hat{X}(0)$

$\quad 49 \qquad \frac{1}{2} \cdot 100$

EEL 6545 Random Processes Final, Dec. 3, 2003. Open book & notes. Two hours.

1. (13 points) Starting *now*, a coin is flipped every hour. If it is heads on the first flip, a marker is place at x = +1; if it is tails, the marker is placed at x = -1. At the second flip, it is moved one unit to the right if the coin reads heads, and 1 to the left if it reads tails; but then it is shifted one-third of the way back to the origin. (So it's true position is either x = 0, 2/3, or -2/3.) The position of the coin is measured to be +0.7, with a measuring device that is unbiased but has an error standard deviation of .5. What is the least-squares estimate for the position of the marker?

-4 -3 -2 -1 0 1 2 3 4

First flip:

$$\left.\begin{array}{l} H \rightarrow X = +1 \\ T \rightarrow X = -1 \end{array}\right\}$$

$$\hat{X}(0) = 0$$

$$E[\hat{X}(0)] = \tfrac{1}{2}(1) + \tfrac{1}{2}(-1)$$

$$\sigma_0^2 = \tfrac{1}{2}(1)^2 + \tfrac{1}{2}(-1)^2 = 1$$

Second Flip:

$$\left.\begin{array}{l} H \text{ move } +1 \\ T \text{ move } -1 \end{array}\right\} \text{and then } \tfrac{1}{3} \text{ of value}$$

$$\Rightarrow X(1) = \begin{array}{l} 2/3 \\ 0 \\ -2/3 \end{array}$$

Model Equation

$$X(1) = \frac{X(0) \pm 1}{3}$$

$$X(1) = \frac{1}{3}X(0) \pm \frac{1}{3}$$

Measurement after 2nd flip:

$$Y(1) = .7 \pm .5$$

$$\hat{X}(1) = K\,Y(1) + L A_0 \hat{X}(0)$$

(will see)

$$= A_0 \hat{X}(0) + K\left[Y(1) - A_0 \hat{X}(0)\right]$$

UNBIASED

$$\mathcal{E}[\hat{X}(1)] = \mathcal{E}[X(1)]$$

$$= K \underbrace{\mathcal{E}[Y(1)]}_{=\mathcal{E}[X(1)]} + L \underbrace{\underbrace{\mathcal{E}[A_0 \hat{X}(0)]}_{= A_0 \underbrace{\mathcal{E}[\hat{X}(0)]}_{=\mathcal{E}[X(0)]}}}_{=\mathcal{E}[X(1)]}$$

Since $X(1) = A_0 X(0) + w_1^{mod}$,

$$\mathcal{E}[X(1)] = A_0\,\mathcal{E}[X(0)] + \underbrace{\mathcal{E}[w_1^{mod}]}_{0}$$

$$\mathcal{E}[\hat{X}(1)] = (K+L)\,\mathcal{E}[X(1)]$$

so $K+L=1$, $L=1-K$

and therefore

$$\hat{X}(1) = A_0 \hat{X}(0) + K\left[Y(1) - A_0 \hat{X}(0)\right]$$

$$e_1^{err} = \hat{X}(1) - X(1) \qquad \mathcal{E}\left[e_1^{err}\right] = 0$$

$$\mathcal{E}\left[e_1^{err\,2}\right] =$$

$$\mathcal{E}\left[\left(A_0 \hat{X}(0) - X(1) + K\left[Y(1) - A_0 \hat{X}(0)\right]\right)^2\right]$$

$$= \mathcal{E}\left[(\alpha + K\beta)^2\right]$$

$$= \mathcal{E}\left[\alpha^2 + 2K\alpha\beta + K^2\beta^2\right]$$

$$= \mathcal{E}\left[\alpha^2\right] + 2K\mathcal{E}\left[\alpha\beta\right] + K^2\mathcal{E}\left[\beta^2\right]$$

$$= A + 2KB + CK^2$$

At the minimum,

$$\frac{d}{dK}\mathcal{E}\left[e_2^{err\,2}\right] = 0 = 2B + 2CK$$

$$\Rightarrow K_{min} = -B/C$$