Why is $\displaystyle |H(\omega)|^2 = \frac{PSD(\text{output})}{PSD(\text{input})}$?

Def'n of transfer function:

$$LX = f$$

If $X = H(\omega)e^{i\omega t}$ & $f = e^{i\omega t}$,

↑ this is the transfer function.

Facts about Four Tranf.

$$1 = \tilde{\delta}(t) = \int_{-\infty}^{\infty} \delta(t)\, e^{-i\omega t}\, dt \qquad f \equiv \frac{\omega}{2\pi}$$

$$\delta(t) = \int_{-\infty}^{\infty} 1\, e^{+i 2\pi f t}\, df$$

Transfer function H, with the factor $e^{i\omega t}$, is the response to $e^{i\omega t}$

$Lx = f, \quad L[H(\omega)e^{i\omega t}] = e^{i\omega t}$

"impulse response" is the response to an impulse:

$$L\, x_{impulse}(t) = \delta(t)$$

Fact: $H(\omega)$ is the ~~F~~ Four. Tran. of $x_{impulse}$.

Why?

$$L\, x_{impulse} = \delta(t)$$

linear $\longrightarrow$ $L \underbrace{\int_{-\infty}^{\infty} \hat{x}(\omega) e^{i\omega t}\, df}_{''} = \int_{-\infty}^{\infty} 1\, e^{i\omega t}\, df$

$$\int_{-\infty}^{\infty} [L\, \hat{x}(\omega) e^{i\omega t}]\, df = \int_{-\infty}^{\infty} e^{i\omega t}\, df$$

So $L\, \underbrace{\hat{x}(\omega) e^{i\omega t}}_{} = e^{i\omega t}$

this is the response to $e^{i\omega t}$, so

$\hat{x}(\omega)$ is the transf. fun. $H(\omega)$.

SO rewrite $x_{impulse}(t)$ as "$h(t)$".

F.T of $h(t)$ is $H(\omega)$

$\uparrow$ impulse ~~func~~ response $\qquad$ $\uparrow$ Transfer fun

$L\, h = \delta(t) \qquad L\, H(\omega) e^{i\omega t} = e^{i\omega t}$

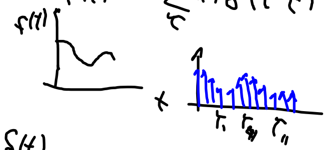
Fact: $Lx = f$ $\quad L\,h(t) = \delta(t)$

$$\Rightarrow x(t) = \int_{-\infty}^{\infty} f(\tau)\, h(t-\tau)\, d\tau$$

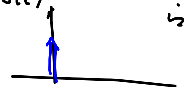
Physical proof.

$f(t) \longrightarrow$ smoothing bowing action on a violin.

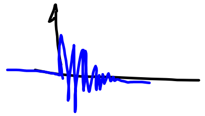
high speed photo. $\Longrightarrow f(t) = \sum_{\tau} a(t)\,\delta(t-\tau)$



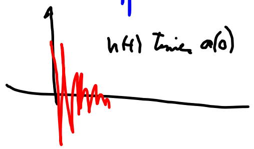
response to $\delta(t)$



is $h(t)$

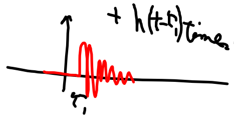


response to $f(t)$ is

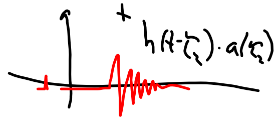


$h(t)$ times $a(0)$

$+\ h(t-\tau_1)$ times $a(\tau_1)$



$+\ h(t-\tau_2)\cdot a(\tau_2)$



$x(t) = \sum_{\tau} a(\tau)\, h(t-\tau)$ when $f(t) = \sum a(\tau)\,\delta(t-\tau)$.

$$x(t) = \int_{-\infty}^{\infty} f(\tau)\, h(t-\tau)\, d\tau \Longleftarrow f(t) = \int_{-\infty}^{\infty} f(\tau)\, \delta(t-\tau).$$

Fact: $\quad L x = f \qquad L h(t) = \delta(t)$

$$\Rightarrow x(t) = \int_{-\infty}^{\infty} f(\tau)\, h(t-\tau)\, d\tau$$

Math proof

$$L x = L \int_{-\infty}^{\infty} f(\tau)\, h(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} f(\tau)\, \underbrace{L\, h(t-\tau)}_{\delta(t-\tau)}\, d\tau$$

$$= f(t)$$

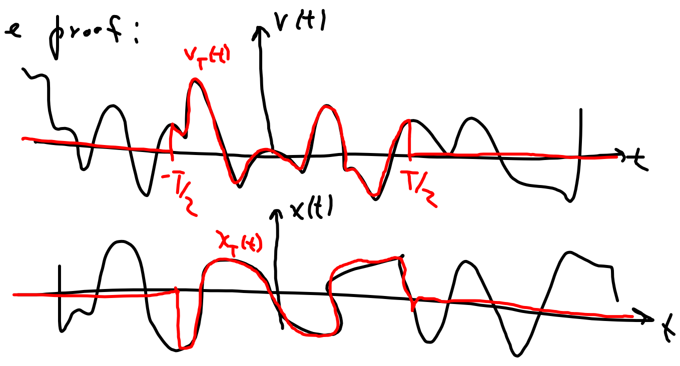
$$x(t) = \int_{-\infty}^{\infty} f(\tau)\, h(t-\tau)\, d\tau$$

Four Trans $\quad \boxed{X(\omega) = F(\omega)\, H(\omega)}$

SO: why is $PSD_{x(t)} = |H(\omega)|^2 \, PSD_{v(t)}$

$Lx = v$ ~~...~~ v random. ?

False proof:



$L\,x(t) = v(t)$

truncate $L\,x_T(t) = v_T(t)$

not quite true: x(t) continues to reverberate, $x(t) \neq x_T(t)$.

accept this:

$$x_T(t) = \int_{-\infty}^{\infty} v_T(\tau)\, h(t-\tau)\, d\tau$$

$$\hat{x}_T(\omega) = \hat{V}_T(\omega)\, H(\omega)$$

$$\frac{|\hat{x}_T(\omega)|^2}{T} = \frac{|\hat{V}_T(\omega)|^2}{T} \, |H(\omega)|^2$$

$$PSD_{x(t)} = PSD_{v(t)} \, |H(\omega)|^2$$

True Proof.

$$L\,x(t) = v(t)$$

$$\mathcal{E}\left[x(0)\,L\,x(t)\right] = \mathcal{E}\left[x(0)\,v(t)\right]$$

$$\parallel$$

$$L\;\mathcal{E}\left[x(0)\,x(t)\right]$$

$$\parallel$$

$$L\,R_x(t) = \mathcal{E}\left[x(0)\,v(t)\right]$$

$$R_x(t) = \int_{-\infty}^{\infty} \underbrace{\mathcal{E}\left[x(0)\,v(\tau)\right]}_{\mathcal{E}\left[v(\tau)\int_{-\infty}^{\infty} v(\tau')\,h(0-\tau')\,d\tau'\right]} h(t-\tau)\,d\tau$$

$$\underbrace{\mathcal{E}\left[v(\tau)\,v(\tau')\right]}_{R_v(\tau-\tau')} h(-\tau')\,d\tau'$$

$$R_x(\tau) = \int_{-\infty}^{\infty} \boxed{\int_{-\infty}^{\infty} R_v(\tau-\tau')\,h(-\tau')} \, h(t-\tau)\,d\tau'\,d\tau$$

$\underline{F^1}$

$$PSD_x(f) = \underbrace{FT(\text{this})}_{FT[R_v]\cdot FT(h(-\tau'))} \cdot H(\omega)$$

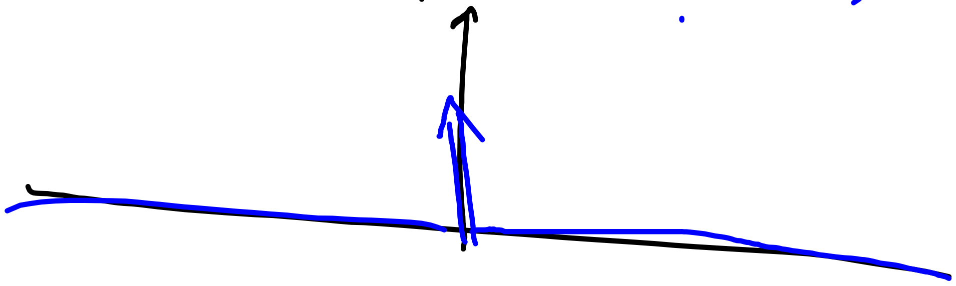
$$FT[R_v]\cdot \underbrace{FT(h(-\tau'))}_{H(\omega)^*} \cdot H(\omega)$$

$$\cancel{PDS}\;PSD_x = PSD_v \cdot |H(\omega)|^2$$

White noise — no autocorrelation.

$$\mathcal{E}[v(t)\, v(t')] = 0 \quad \text{if } t \neq t'$$

$$R_x(\tau) = \sigma^2 \delta(\tau)$$



$$R_x(0) = \mathcal{E}[X(t)X(t)] = \mathcal{E}[X(t)^2] = \sigma^2$$

$$FT\{R_x(\tau)\} = PSD = \int_{-\infty}^{\infty} R_x(\tau)\, e^{-i\omega\tau}\, d\tau = \sigma^2$$