Difference Eqs.   ARMA

$$x(n) = \overbrace{b(0)v(n) + b(1)v(n-1) \cdots}^{MA}$$

$$\underbrace{- a(1)x(n-1) - a(2)x(n-2)}_{AR} \cdots$$

Express $x(n)$ directly in terms of input $v(\cdot)$

MA

$$x(n) = b(0)\,v(n) + b(1)\,v(n-1) + b(2)\,v(n-2) \ldots$$

+ express x's in terms of v's

$$x(n) = \_\_\,v(n) + \_\_\,v(n-1) + \_\_\,v(n-2) + \ldots$$

We've got it already: $b(0)$ $b(1)$ $b(2)$

For ARMA

$$x(n) = b(0)\,v(n) + b(1)\,v(n-1) + b(2)\,v(n-2)$$

$$- a(1)\,x(n-1)$$

I want

$$x(n) = \_\_\_\, v(n) + \_\_\_\, v(n-1) + \_\_\_\, v(n-2) + \_\_\_\, v(n-3)$$

$$+ \_\_\_\, v(n-4)$$

Substitution: $x(n) = b(0)\,v(n) + b(1)\,v(n-1) + b(2)\,v(n-2)$

$$- a(1)\left[b(0)\,v(n-1) + b(1)\,v(n-2) + b(2)\,V(n-3) \right.$$

$$\left. - a(1)\quad x(n-2) \right]$$

Substitution: $x(n) = b(0)v(n) + b(1)v(n-1) + b(2)V(n-2)$

$- a(1)\Big[b(0)v(n-1) + b(1)v(n-2) + b(2)V(n-3)$

$- a(1)\Big\{ b(0)v(n-2) + b(1)v(n-3)$

$+ b(2)V(n-4) - a(1)x(n-3)\Big\}\Big]$

$$x(n) = b(0)V(n) + \{b(1) - a(1)b(0)\}V(n-1)$$

$$+ \left[b(2) - a(1)b(1) + a(1)^2 b(0)\right]V(n-2)$$

$$+ \underline{\qquad\qquad\qquad} v(n-3) + \underline{\qquad\qquad} V(n-4) + \ldots$$

Summary: The solution to

$$x(n) = b(0)v(n) + b(1)v(n-1) + \cdots$$
$$- a(1)x(n-1) - a(2)x(n-2) - \cdots$$

can be written

$$x(n) = h(0)v(n) + h(1)v(n-1) + h(2)v(n-2) + h(3)v(n-3) \ldots$$

For MA: sum is finite & $h(0) = b(0)$, $h(1) = b(1)$, $h(2) = b(2)$

~~F~~For ARMA, sum is infinite, &

$$h(0) = b(0)$$
$$h(1) = b(1) - a(1)b(0)$$
$$h(2) = b(2) - a(1)b(1) + a(1)^2 b(0)$$
$$h(3) \ldots$$

~~ARMA modeling problems.~~

Yule-Walker equations, Assume v is i.i.d. noise, zero mean.

$$x(n) + a(1)\,x(n-1) + a(2)\,x(n-2) \ldots = b(0)\,v(n) + b(1)\,v(n-1) \ldots$$

multiply by $x(n-k)$ & $\mathcal{E}[\_\_\_]$.

LHS: $\mathcal{E}[\,x(-)\,x(-)\,] = R_x(\text{difference})$

RHS: $\mathcal{E}[\,x(-)\,v(-)\,]$ (?)

$\downarrow$

express in terms $b(0)v(n)+b(1)v(n-1)+\ldots$

RHS: $\mathcal{E}[\,v(-)\,v(-)\,] = \begin{cases} 0 & \text{unless times match} \\ \sigma_v^2 & \end{cases}$

Yule-Walker equations (see notes).

Problem 1: given ARMA model, ~~?~~ a's & b's ;

FIND the autocorrelation.

EASY: solve the Y-W eqs for $R_x(n)$.

Problem 2: given autocorr $R_x(n)$, FIND the ARMA model

NOT SO EASY:

First guess p, q (the "order" of the model).

$$\underbrace{a(1), \ldots, a(p),\ b(0), \ldots b(q)}_{p+q+1} \text{ unknown.}$$

Solve the first $(p+q+1)$ Y-W eqs. for a's & b's;

then check the remaining Y-W eqs.

If they don't work, increase p & q & start again.

LINEAR SYSTEMS THEORY: GENERAL CONSIDERATIONS.

Difference Eq.

$\Delta E \quad x(n) + a(1)\, x(n-1) + a(2)\, x(n-2) = b(0)\, v(n)$

Differential Eq. $\ddot{x}(t) + \nu \dot{x}(t) + K(x) = f(t)$

DE

Integral Eq. $\int_a^b K(\tau, t)\, x(\tau)\, d\tau = f(t)$

TRANSFER FUNCTION $H(\omega)$, $\omega = 2\pi f$

$x(t) = H(\omega) e^{i\omega t}$ is the solution to the system when RHS $= e^{i\omega t}$ (v, f).

~~Example~~ DE

$$\ddot{x} + 2\dot{x} + 3x = f(t)$$

$$-\omega^2 H e^{i\omega t} + 2 i\omega H e^{i\omega t} + 3 H e^{i\omega t} = e^{i\omega t}$$

$$H(\omega) = \frac{1}{-\omega^2 + 2i\omega + 3}$$

Example. ΔE

$$x(n) = b(0)\, v(n) - a(1)\, x(n-1) - a(2)\, x(n-2)$$

$X(n) = H(\omega) e^{i\omega n}$

$$H(\omega) e^{i\omega n} = b(0) e^{i\omega n} - a(1) H e^{i\omega(n-1)} - a(2) H e^{i\omega(n-2)}$$

$$H(\omega) = \frac{b(0)}{1 + a(1) e^{-i\omega} + a(2) e^{-i2\omega}}$$

FACT linear system

$$L\,x = v$$

with transfer function $H(\omega)$

then PSD's are related by

$$S_x(\omega) = |H(\omega)|^2 S_v(\omega)$$