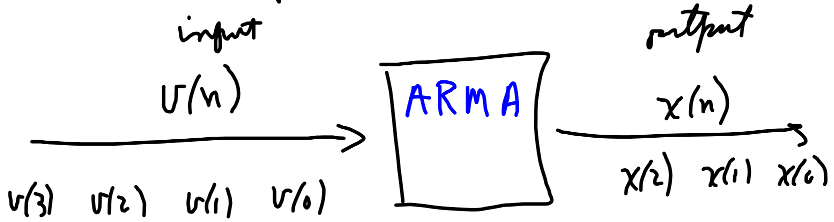
Discrete processes.



Difference Equation

$$x(n) = \underbrace{b(0)v(n) + b(1)v(n-1) + b(2)v(n-2)}_{\text{Moving Average}} \underbrace{- a(1)x(n-1) - a(2)x(n-2)}_{\text{Auto regression}} \dots$$

$$\sum_{l=0}^{p} a(l)\, x(n-l) = \sum_{l=0}^{q} b(l)\, v(n-l)$$

$a(0) = 1$

In random process modeling, $v(n)$ are i.i.d. random variables: $\{v\}$ is white noise.

$\boxed{\text{zero-mean}}$

Let $(1)\; x(n) = \left(\frac{1}{2}\right) x(n-1) + (1) v(n)$ ← AR

Question 1. Given a measured random process ~~x(t)~~ x(n), find an ARMA model which could have generated it.

First (use Bartlett ~~to~~) extract the autocorrelation $R_x(\tau)$.

Then find {a's, b's} which give that autocorrelation.

Second question, given ARMA a's & b's,

what is $R_X(\tau)$ ~~out~~ of the output?

The "Yule-Walker" relate a's, b's, R_X's.

Use them to answer question 1 or question 2,

Yule-Walker Equations.

$p=3$

$$a(0)x(n) + a(1)x(n-1) + a(2)x(n-2) + a(3)x(n-3)$$

$q=2$

$$= b\,v(n) + b(1)v(n-1) + b(2)v(n-2)$$

(3,2) ARMA process.

$$\sum_{l=0}^{p} a(n)\,x(n-l) = \sum_{l=0}^{q} b(n)\,v(n-l)$$

To get Y-W eqs, multiply by $x(n-k)$ & take $E\{-\}$.

$$\sum_{l=0}^{p} a(n)\,\underbrace{E\{x(n-l)\,x(n-k)\}}_{R_x(k-l)} = \sum_{l=0}^{q} b(n)\,\underbrace{E\{v(n-l)\,x(n-k)\}}_{\substack{\text{"cross correlations"}\\ R_{vx}(k-l)}}$$

$$\mathcal{E}\left\{ v(n-\ell)\, x(n-k) \right\}$$

~~X~~ Suppose $n-K < n-\ell$ (late noise, early output)

$$\mathcal{E}\left\{ v(\text{late})\, x(\text{early}) \right\} = 0$$

because $v(\text{late})$ is independent of $x(\text{early})$.

But $x(\text{late})$ depends on $v(\text{early})$, so not independent.

To get $\mathcal{E}\{x(\text{late})\, v(\text{early})\}$, we need to express $x(n)$ in terms of $v(n), v(n-1), v(n-2), \ldots$

We need

$$x(n) = h(0)\,v(n) + h(1)\,v(n-1) + h(2)\,v(n-2) + h(3)\,v(n-3)$$

$$\ldots$$

$$\Rightarrow \mathcal{E}\{x(n)\, v(n-r)\} = \mathcal{E}\{[h(0)v(n) + h(1)v(n-1) + \cdots$$
$$\cdots]\,v(n-r)\}$$

$$= \begin{cases} h(r)\,\underbrace{\mathcal{E}\{v(n-r)^2\}}_{\sigma_v^2} & \text{if } r \geq 0 \\ 0 & \text{if } r < 0 \end{cases}$$

$$\boxed{\begin{array}{l} \text{from next page} \Rightarrow \quad n-k \qquad n-l \\ r = l-k \\ l-k \geq 0 \Leftrightarrow l \geq k \end{array}}$$

$$\mathcal{E}\{v(i)\,v(j)\} = \begin{cases} 0, \text{ by independence + zero-mean, if } i \neq j \\ \sigma_v^2 \quad \text{if } i = j \end{cases}$$

$$\sum_{l=0}^{p} a(l)\, x(n-l) = \sum_{l=0}^{q} b(l)\, v(n-l)$$

$\cdot\, x(n-k)$ & take $\mathcal{E}\{\text{———}\}$.

$$\sum_{l=0}^{p} a(l)\, R_x(k-l) = \sum_{l=0}^{q} b(l)\, \mathcal{E}\{x(n-k)\, v(n-l)\}$$

from previous page,
$h(l-k)\sigma_v^2$ if $l \geq k$,
0 otherwise

Y-W $\displaystyle\sum_{l=0}^{p} a(l)\, R_x(k-l) = \sum_{l=k}^{q} b(l)\, h(l-k)\, \sigma_v^2$

$$\text{Y-W} \quad \sum_{l=0}^{p} a(l)\, R_x(k-l) = \sum_{l=k}^{q} b(l)\, h(l-k)\, \sigma_v^2$$

$$\begin{bmatrix} R_x(0) & R_x(-1) & R_x(-2) & R_x(-3) \\ R_x(1) & R_x(0) & R_x(-1) & R_x(-2) \\ & & & \\ & & & \end{bmatrix} \begin{bmatrix} a(0) = 1 \\ a(1) \\ a(2) \\ a(3) \end{bmatrix} = \begin{bmatrix} b(0)\, h(0)\, \sigma_v^2 + b(1)\, h(1)\, \sigma_v^2 + b(2)\, h(2)\, \sigma_v^2 + b(3) \text{ No!} \\ b(1)\, h(0)\, \sigma_v^2 + b(2)\, h(1)\, \sigma_v^2 + b(3) \text{ No!} \end{bmatrix}$$

$(k=0)$

$q=2$

$(k=1)$

See notes for elaboration.

How do you find the h's ?

$q=0$ $\quad p=2$

$$x(n) = b(0)\,v(n) - a(1)\,x(n-1) - a(2)\,x(n-2)$$

$$x(0) = b(0)\,v(0) - a(1)\cdot 0 - a(2)\cdot 0$$

$$x(1) = b(0)\,v(1) - a(1)\,x(0) - a(2)\,x(-1)$$

$$= b(0)\,v(1) - a(1)\,b(0)v(0) - 0$$

$$x(2) = b(0)\,v(2) - a(1)\left[b(0)\,v(1) - a(1)\,b(0)\,v(0)\right] - a(2)\,b(0)\,v(0)$$

and so on.

Compare with

$$\underbrace{x(n)}_{} = \underbrace{h(0)}_{=b(0)}\,v(n) + \underbrace{h(1)}_{=-a(1)b(0)}\,v(n-1) + \underbrace{h(2)}_{=a(1)^2 b(0) - a(2)b(0)}\,v(n-2) + \cdots$$