Kalman filter.

underlying process is

$$d(n) = a(n-1)\, d(n-1) + v_1(n)$$

$$\mathcal{E}[d(n)] = a(n-1)\, \mathcal{E}[d(n-1)] + 0$$

measurement process

$$x(n) = d(n) + v_2(n)$$

$$\mathcal{E}[x(n)] = \mathcal{E}[d(n)] + 0$$

Kalman filter

$$\widehat{d(n)} = a(n-1)\, \widehat{d(n-1)} + K(n)\left[x(n) - a(n-1)\hat{d}(n-1)\right]$$

Kalman gain

check bias

$$\mathcal{E}[\hat{d}(n)] = \mathcal{E}[d(n)] = a(n-1)\, \mathcal{E}[d(n-1)]$$

desired

$$\hat{d}(n) = a(n-1)\,\hat{d}(n-1) + K(n)\left[x(n) - a(n-1)\,\hat{d}(n-1)\right]$$

Expected Value

$$E\left[\hat{d}(n)\right] = \underbrace{a(n-1)\,\underbrace{E\left[\hat{d}(n-1)\right]}_{E[d(n-1)]}}_{E[d(n)]} + K(n)\left[\underbrace{E\left[x(n)\right]}_{E[d(n)]} - \underbrace{a(n-1)\,\underbrace{E\left[\hat{d}(n-1)\right]}_{E[d(n-1)]}}_{E[d(n)]}\right]$$

$$\hat{d}(n) = a(n-1)\hat{d}(n-1) + K(n)\left[x(n) - a(n-1)\hat{d}(n-1)\right]$$

$$\text{Error} = \hat{d}(n) - d(n) = e_n$$

$$e_n^2 = \left[\hat{d}(n) - d(n)\right]^2 = \left[a(n-1)\hat{d}(n-1) + K(n)\left[x(n) - a(n-1)\hat{d}(n-1)\right] - d(n)\right]^2$$

$$= \left[a(n-1)\hat{d}(n-1) + K(n)\left[d(n) - a(n-1)\hat{d}(n-1)\right] + K(n)V_2(n) - \left[a(n-1)d(n-1) + V_1(n)\right]\right]^2$$

$$= \left[a(n-1)\underbrace{\left(\hat{d}(n-1) - d(n-1)\right)}_{e_{n-1}}\right.$$

$$\left. + K(n)\left[d(n) - a(n-1)\hat{d}(n-1)\right] + K(n)V_2(n) - V_1(n)\right]^2$$

$$d(n) = a(n-1)d(n-1) + V_1(n) \qquad \hat{d}(n-1) = d(n-1) + e_{n-1}$$

$$= \left[a(n-1)e_{n-1} + K(n)\left(V_1(n) - a(n-1)e_{n-1}\right) + K(n)V_2(n) - V_1(n)\right]^2$$

$$\mathcal{E}\left[e_n^2\right] =$$

$$= \mathcal{E}\left[a(n-1)\, e_{n-1} + K(n)\left(v_1(n) - a(n-1)\, e_{n-1}\right) + K(n)\, v_2(n) - v_1(n)\right]^2$$

$$= \mathcal{E}\left[a(n-1)\, e_{n-1} + (K(n)-1)\, v_1(n) + K(n)\, v_2(n) - K(n)\, a(n-1)\, e_{n-1} \right]^2$$

$$(1-K(n))\, a(n-1)\, e_{n-1}$$

$$\mathcal{E}(e_n^2) = [1-K(n)]^2\, a(n-1)^2\, \mathcal{E}(e_{n-1}^2) + (K(n)-1)^2\, \sigma_{v_1}^2 + K(n)^2\, \sigma_{v_2}^2$$

$$+ \text{terms like } \mathcal{E}\left[e_{n-1}\, v_1(n)\right]$$

yesterday's error   today's noise ← independent

$$\mathcal{E}(e_n^2) = [1-K(n)]^2 a(n-1)^2 \mathcal{E}(e_{n-1}^2) + (K(n)-1)^2 \sigma_{v_1}^2 + K(n)^2 \sigma_{v_2}^2$$

$$= [1-K(n)]^2 \left\{ a(n-1)^2 \mathcal{E}(e_{n-1}^2) + \sigma_{v_1}^2 \right\} + K(n)^2 \sigma_{v_2}^2$$

$$\frac{\partial \{\text{this}\}}{\partial K(n)} = \{\text{——}\}\, 2[1-K(n)](-1) + 2K(n)\sigma_{v_2}^2 = 0$$

$$K(n) = \frac{\{\text{——}\}}{\{\text{——}\} + \sigma_{v_2}^2}.$$

$$K(n) = \frac{\left\{ a(n-1)^2 \mathcal{E}(e_{n-1}^2) + \sigma_{v_1}^2 \right\}}{\left\{ a(n-1)^2 \mathcal{E}(e_{n-1}^2) + \sigma_{v_1}^2 \right\} + \sigma_{v_2}^2}$$

Problem: we need to update $\mathcal{E}(e_n^2)$ so that, tomorrow, we can compute $K(n+1)$.

Solution

$$\mathcal{E}(e_n^2) = [1-K(n)]^2 a(n-1)^2 \mathcal{E}(e_{n-1}^2) + (K(n)-1)^2 \sigma_{v_1}^2 + K(n)^2 \sigma_{v_2}^2$$

Note:

If you have $\mathcal{E}(e_0^2)$, $\{a(n)\}$, $\sigma_{v_1}^2$, $\sigma_{v_2}^2$; you can compute <u>all</u> the Kalman gain $K(1), K(2), K(3), \ldots$ without looking at the measurement data $x(n)$. !!!!