~~Today~~ Today's problem: $x(n) = d(n) + v(n)$

estimate $d(n)$ from $x(n)$ & $x(n-1)$

$$\hat{d}(n) = w(0)\,x(n) + w(1)\,x(n-1)$$

Least Mean Square ~~Est~~ Estimator.

LMS Estim.

$$\text{Error} = \hat{d}(n) - d(n) = w(0)\,x(n) + w(1)\,x(n-1) - d(n)$$

$$E\left[|\text{Error}|^2\right] = E\left[|\hat{d}(n)|^2\right] + E\left[|d(n)|^2\right] - 2E\left[\hat{d}(n)\,d(n)\right]$$

$$= E\left\{w(0)^2 x(n)^2 + w(1)^2 x(n-1)^2 + 2w(0)\,w(1)\,x(n)\,x(n-1)\right\}$$

$$+ r_D(0) - 2E\left\{\left(w(0)\,x(n) + w(1)\,x(n-1)\right)d(n)\right\}$$

$$E\left[|Error|^2\right] = E\left[\hat{d}(n)^2\right] + E\left[d(n)^2\right] - 2E\left[\hat{d}(n)\, d(n)\right]$$

$$= E\left\{ w(0)^2 x(n)^2 + w(1)^2 x(n-1)^2 + 2w(0)w(1)\,x(n)\,x(n-1) \right\}$$

$$+ \; r_D(0) \; - 2E\left\{ \left(w(0)x(n) + w(1)x(n-1)\right) d(n) \right\}$$

$$= w(0)^2 r_x(0) + w(1)^2 r_x(0) + 2w(0)w(1)\, r_x(1)$$

$$+ r_d(0) - 2w(0)\, r_{xD}(0) - 2w(1)\, r_{xD}(-1)$$

$$r_{xD}(m) = E\left\{ x(n+m)\, D(n) \right\}$$

$$E\left[x(\ell)\,x(m)\right] = r_x(\ell - m) = r_x(m-\ell)$$

$$E\left[x(\ell)\, d(m)\right] \doteq r_{xD}(\ell - m)$$

$$E\left[x(m)\, d(\ell)\right] = r_{xD}(m-\ell)$$

$$E\left[x(n-1)\, d(n)\right] = r_{xD}(-1)$$

$$\parallel$$

$$E\left[d(n)\, x(n-1)\right] = r_{Dx}(+1)$$

To match my YulWalWienKal.doc notes,

$$MSE = w(0)^2\, r_x(0) + w(1)^2\, r_x(0) + 2\, w(0) w(1)\, r_x(1)$$

$$- 2\, w(0)\, r_{dx}(0) - 2\, w(1)\, r_{dx}(1) + r_d(0)$$

To set "LEAST" MS error, minimize:

$$0 = \frac{\partial\, MSE}{\partial\, w(0)} = 2 w(0) r_x(0) + 2 w(1) r_x(1) - 2 r_{dx}(0)$$

$$0 = \frac{\partial\, MSE}{\partial\, w(1)} = 2 w(1) r_x(0) + 2 w(0) r_x(1) - 2 r_{dx}(1)$$

Wiener-Hopf Equations

$$\begin{bmatrix} r_x(0) & r_x(1) \\ r_x(1) & r_x(0) \end{bmatrix} \begin{bmatrix} w(0) \\ w(1) \end{bmatrix} = \begin{bmatrix} r_{dx}(0) \\ r_{dx}(1) \end{bmatrix}$$

How do get ~~get~~ $r_x(-)$, $r_{dx}(-)$?

Use model:

$$x(n) = d(n) + v(n)$$

$$r_{dx}(k) = E\{ d(n) \underbrace{x(n-k)}_{d(n-k)+v(n-k)} \} =$$

$$= \underbrace{E\{d(n)\,d(n-k)\}}_{= r_d(k)} + \underbrace{E\{d(n)\,v(n-k)\}}_{= E\{d(n)\}E\{v(n-k)\} = 0}$$

$$r_{dx}(k) = r_d(k)$$

for this model

not always!

$$x(n) = d(n) + v(k)$$

$$r_x(k) = E\left\{ x(n)\, x(n-k) \right\}$$

$$= E\left\{ \big(d(n) + v(n)\big)\big(d(n-k) + v(n-k)\big) \right\}$$

$$= r_d(k) + 0 + 0 + \begin{cases} 0 & \text{if } k \neq 0 \\ \sigma_v^2 & \text{if } k = 0 \end{cases}$$

$$r_x(k) = r_d(k) + \sigma_v^2 \delta_{k0}$$

$$\begin{bmatrix} r_x(0) & r_x(1) \\ r_x(1) & r_x(0) \end{bmatrix} \begin{bmatrix} w(0) \\ w(1) \end{bmatrix} = \begin{bmatrix} r_{dx}(0) \\ r_{dx}(1) \end{bmatrix}$$

$$\begin{bmatrix} r_d(0) + \sigma_v^2 & r_d(1) \\ r_d(1) & r_d(0) + \sigma_v^2 \end{bmatrix} \begin{bmatrix} w(0) \\ w(1) \end{bmatrix} = \begin{bmatrix} r_d(0) \\ r_d(1) \end{bmatrix}$$

New notes:

$d(n)$ is ARMA $(1,0)$.

assume $b(0)=1$ (incorporate it into $v(n)$)

$$d(n) = -a(1)\,d(n-1) + v_1(n)\,b(0)$$

$\Longrightarrow$ get $r_d(n)$, PSD_d, interpret Wiener filter:

Y-W
$$\begin{bmatrix} r_d(0) & r_d(1) \\ r_d(1) & r_d(0) \\ r_d(2) & r_d(1) \\ \vdots & \end{bmatrix} \begin{bmatrix} 1 \\ a(1) \end{bmatrix} = \begin{bmatrix} b^2(0)\,\sigma_{v_1}^2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$r_d(0) + r_d(1)\,a(1) = \sigma_{v_1}^2$$

$$r_d(1) + r_d(0)\,a(1) = 0 \quad \Rightarrow r_d(1) = -a(1)\,r_d(0)$$

$$\Rightarrow r_d(0) - a(1)\,r_d(0)\,a(1) = \sigma_{v_1}^2$$

$$r_d(0) = \frac{\sigma_{v_1}^2}{1-a(1)^2}$$

$$r_d(1) = \left(\text{same}\right)\left[-a(1)\right]$$

$$r_d(2) = \left(\quad\right)\left[-a(1)\right]^2$$

$$r_d(3) = \left(\quad\right)\left[-a(1)\right]^3$$

$$\vdots$$

$$r_d(n) = \frac{\sigma_{v_1}^2}{1-a(1)^2}\left[-a(1)\right]^n \quad \text{for } n = 0,1,2,3,\dots$$

$$r_d(n) = r_d(|n|) \quad \text{if } n<0$$

generally, $$r_d(n) = \frac{\sigma_v^2}{1-a(1)^2}\left[-a(1)\right]^{|n|}$$

PSD = Four Trans of $\{ r_d(n) \}$

$$= \sum_{n=-\infty}^{\infty} r_d(n)\, e^{-i\omega n}$$

$$\frac{\sigma_v^2}{1 - a(1)^2} \sum_{n=G}^{\infty} \{-a(1)\}^{|n|} e^{-i\omega n}$$

$$\left(\quad '' \quad \right) \left\{ \sum_{n=0}^{\infty} \alpha^n + \sum_{n=-1}^{-\infty} \beta^{-n} \right\}$$

$$\alpha = -a(1) e^{-i\omega}$$

$$\beta = -a(1) e^{i\omega}$$

$$\sum_{n=0}^{\infty} \alpha^n = 1 + \alpha + \alpha^2 + \alpha^3 + \alpha^4 + \cdots$$

Times $(1-\alpha)$

$$\begin{array}{r} 1 - \alpha \\ \hline 1 + \alpha + \alpha^2 + \alpha^3 + \alpha^4 + \cdots \\ - \alpha - \alpha^2 - \alpha^3 - \alpha^4 - \cdots \\ \hline \end{array}$$

$$\left(\sum_{n=0}^{\infty} \alpha^n\right)(1-\alpha) = 1$$

$$\text{So} \quad \sum_{n=0}^{\infty} \alpha^n = \frac{1}{1-\alpha}$$

$$\sum_{n=-1}^{-\infty} \beta^{-n} \equiv \sum_{n=1}^{\infty} \beta^{n} = \sum_{n=0}^{\infty} \beta^{n} - \beta^{0}$$

$$= \frac{1}{1-\beta} - 1$$

$$\frac{\sigma_{v_1}^2}{1-a(1)^2}\sum_{n=-\infty}^{\infty}\left[-a(1)\right]^{|n|} e^{-i\omega n}$$

$$\left(\quad " \quad \right)\left\{ \sum_{n=0}^{\infty} \alpha^n + \sum_{n=-1}^{-\infty} \beta^{-n} \right\}$$

$$\alpha = -a(1)e^{-i\omega} \qquad \beta = -a(1)e^{i\omega}$$

$$PSD = \frac{\sigma_{v_1}^2}{1-a(1)^2}\left\{ \frac{1}{1+a(1)e^{-i\omega}} + \frac{1}{1+a(1)e^{i\omega}} - 1 \right\}$$

$$= \frac{\sigma_{v_1}^2}{1-a(1)^2}\left\{ \frac{2+a(1)\cdot 2\cos\omega}{\left|1+a(1)e^{-i\omega}\right|^2} - 1 \right\}$$

Inverse Fourier Transform:

$$r_d(n) = \frac{1}{2\pi}\int_{-\infty}^{\infty} PSD_d(\omega)\, e^{i\omega t}\, d\omega$$ wrong

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} PSD_d(\omega)\, e^{i\omega n}\, d\omega$$ true for discrete

Take $a(1) = .8$ $\sigma_v^2 = 3$

$$d(n) = .8\, d(n-1) + v(n)$$

$\Longrightarrow$

$$w(0) = .4048$$

$$w(1) = .2381$$

$$\hat{d}(n) = .4048\, x(n) + .2381\, x(n-1)$$

$$d(n) = .8\, d(n-1) + \text{noise}$$

$$x(n) = d(n) + \text{noise}$$

Kalman filter.

Simplest model:

No! $d(n) = \underline{a(1)}\, d(n-1) + v_1(n)$

stationary

$$d(n) = a(n-1)\, d(n-1) + v_1(n)$$

not ~~sd~~ stationary

Elaboration:

if $d(n) = a(1)\, d(n-1) + v_1(n)$

then $d(2) = .8\, d(1) + -$

$d(3) = .8\, d(2) + —$

$d(4) = .8\, d(3) + —$

But $d(n) = a(n-1)\, d(n-1) + v_1(n)$

$d(2) = .8\, d(1) + —$

$d(3) = .9\, d(2) + —$

$d(4) = .7\, d(3) + —$

desired signal $d(n) = a(n-1)\,d(n-1) + v_1(n)$

measured $x(n) = d(n) + v_2(n)$

Kalman estimator

$$\hat{d}(n) = \underline{A}\;\hat{d}(n-1) + \underline{B}\left[x(n) - \underline{A}\;\hat{d}(n-1)\right]$$

Requirement #1 : $\mathcal{E}[\hat{d}(n)] = d(n)$ $\Leftarrow$ Will see that $A = a(n-1)$

" #2 : $\mathcal{E}[\text{error}^2]$ is minimal.

$\Rightarrow$ B is "Kalman gain"

Kalman filter

$$\hat{d}(n) = a(n-1)\,\hat{d}(n-1) + K(n)\left[x(n) - a(n-1)\,\hat{d}(n-1)\right]$$

Biasi

$$E[\hat{d}(n)] = a(n-1)\,E[\hat{d}(n-1)] + K(n)\,E[x(n)] - K(n)\,a(n-1)\,E[\hat{d}(n-1)]$$

assume $E[\hat{d}(n-1)] = d(n-1)$

to show: $E[\hat{d}(n)] = d(n)$

$$= a(n-1)\,d(n-1) + K(n)\,d(n) - K(n)\,a(n-1)\,d(n-1)$$

$$= K(n)\,d(n) + [1 - K(n)]\,\underbrace{a(n-1)\,d(n-1)}_{d(n)\,-\,v(n)}$$