Today's problem: \[ x(n) = d(n) + v(n) \]

Estimate \( d(n) \) from \( x(n) \) and \( x(n-1) \)

\[ \hat{d}(n) = w(0)x(n) + w(1)x(n-1) \]

Least Mean Square Estimator

LMS Estimator

\[ \text{Error} = \hat{d}(n) - d(n) = w(0)x(n) + w(1)x(n-1) - d(n) \]

\[ \sum \left[ |\text{Error}| \right] = \sum \left[ \hat{d}(n) \right]^2 + \sum \left[ d(n) \right]^2 - \sum \left[ \hat{d}(n) d(n) \right] \]

\[ = \sum \left\{ w(0)^2 x(n)^2 + w(1)^2 x(n-1)^2 + 2w(0)w(1)x(n)x(n-1) \right\} \]

\[ + \sum \left[ \hat{d}(0) \right]^2 - 2 \sum \left[ (w(0)x(n) + w(1)x(n-1)) \right] \hat{d}(n) \]
\[ \sum \mathbb{E} \left[ |E_{\text{new}}| \right] = \sum \mathbb{E} \left[ \hat{\theta}_n \right] \mathbb{E} \left[ \| \theta \| \right]^2 - \frac{1}{2} \mathbb{E} \left[ \sum a(n) d(n) \right] \]

\[ = \mathbb{E} \left\{ w(0)^2 x(0)^2 + w(1)^2 x(1)^2 + 2 w(0) w(1) x(1) x(0) \right\} \]

\[ + \mathbb{E} \left\{ \widehat{\varepsilon}_d(o) X \right\} - 2 \mathbb{E} \left\{ (w(n)x(n) + w(n)x(n)) \right\} \mathbb{E} \left\{ \varepsilon_i \right\} \]

\[ = w(0)^2 r(0) + w(1)^2 r(1) + 2 w(0) w(1) r(1) \]

\[ + \mathbb{E} \left\{ \widehat{\varepsilon}_d(o) - 2 w(0) r(0) - 2 w(1) r(1) \right\} \]

\[ r_{x_d}(n) = \mathbb{E} \left\{ X_{(n+m)} D_{(n)}^2 \right\} \]

\[ \mathbb{E} \left[ X(l) X(m) \right] = r_x(l-m) = r_x(m-l) \]

\[ \mathbb{E} \left[ X(l) D(m) \right] = r_{x_d} (l-m) \]

\[ \mathbb{E} \left[ X(m) D(l) \right] = r_{x_p} (m-l) \]

\[ \mathbb{E} \left[ X(n-l) D(n) \right] = r_{x_d} (-l) \]

\[ \mathbb{E} \left[ D(n) X(n-l) \right] = r_{d_x} (41) \]
To match my Yulwal Mian Kel. doe note,

\[ \text{MSE} = w(0)^2 R_x(0) + w(1)^2 R_x(1) + 2 w(0) w(1) R_{dX}(1) \]

\[ -2 w(0) R_{DX}(0) - 2 w(1) R_{dX}(1) + r_d(0) \]

To set \textit{LEAST} MSE error, minimize:

\[ 0 = \frac{2 \text{MSE}}{2 w(0)} = 2 w(0) R_x(0) + 2 w(1) R_x(1) - 2 R_{dX}(0) \]

\[ 0 = \frac{2 \text{MSE}}{2 w(1)} = 2 w(1) R_x(0) + 2 w(0) R_x(1) - 2 R_{dX}(1) \]

\[ \text{Wiener} - \begin{bmatrix} R_x(0) & R_x(1) \\ R_x(1) & R_x(0) \end{bmatrix} \begin{bmatrix} w(0) \\ w(1) \end{bmatrix} = \begin{bmatrix} R_{dX}(0) \\ R_{dX}(1) \end{bmatrix} \]

\[ \text{Hopf Eqaotions} \]
How do we get \( r_x(-) \), \( r_x^d(-) \)?

Use model:

\[
\chi(n) = d(n) + \nu(n)
\]

\[
r_{d_x}(k) = \mathbb{E}\{d(n) \chi(n-k)\} = d(n-k) + \nu(n-k)
\]

\[
= \mathbb{E}\{d(n)d(n-k)\} + \mathbb{E}\{d(n)\nu(n-k)\}
\]

\[
r_{d}(k) = \mathbb{E}\{d(n)\nu(n-k)\} \text{ for this model, not always.}
\]

\[
r_{d_x}(k) = r_{d}(k)
\]
\[ x(n) = d(n) + v(k) \]

\[ r_x(k) = E \left\{ x(n) x(n-k) \right\} \]

\[ = E \left\{ (d(n) + v(n)) (d(n-k) + v(n-k)) \right\} \]

\[ = r_d(k) + 0 + 0 + \begin{cases} 0 & \text{if } k \neq 0 \\ \sigma_v^2 & \text{if } k = 0 \end{cases} \]

\[ r_x(k) = r_d(k) + \sigma_v^2 \delta_{k0} \]
\[ d(n) = ARMA(1, 1) \]
\[ d(n) = -a(1) d(n-1) + v(n) b(n) \]
\[ \Rightarrow \exists \gamma \in \mathbb{R} \text{ PSD filter, intercept white filter:} \]
\[
\begin{bmatrix}
\gamma_x(n) & \gamma_v(n) \\
\gamma_v(n) & \gamma_w(n)
\end{bmatrix}
\begin{bmatrix}
1 \\
a(n)
\end{bmatrix}
= 
\begin{bmatrix}
\delta^2_a \sigma_\epsilon^2 \\
0 \\
0
\end{bmatrix}
\]
\[ \gamma_x(n) + \gamma_v(n) \sigma_v^2 = \sigma_y^2 \]
\[ \gamma_v(n) = 0 \implies \gamma_x(n) = -a(n) \gamma_x(n) \]
\[ \gamma_x(n) = \frac{\sigma_y^2}{1 - a(n)^2} \]
\[ \gamma_v(n) = \left[ a(n) \right] \gamma_v(n) \]
\[ \gamma_w(n) = \left[ a(n) \right]^2 \gamma_v(n) \]
\[ \text{Generally, } \gamma_x(n) = \frac{\sigma_y^2}{1 - a(n)^2} \sum_{k=0}^{n} a(k)^2 \]
\[ \text{PSD} = \frac{\sigma^2}{1 - a(1)^2} \sum_{n=0}^{\infty} a(n) e^{-i\omega n} \]

\[ \alpha = -a(1) e^{-i\omega} \]
\[ \beta = -a(1) e^{i\omega} \]
\[ \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \ldots \]

Term: \((1-x)\)

\[
\frac{1 - x}{1 + x + x^2 + x^3 + x^4 + \ldots}
\]

\[
- x - x^2 - x^3 - x^4 - \ldots
\]

\[
\left( \sum_{n=0}^{\infty} x^n \right) \cdot (1 - x) = 1
\]

\[
\sum_{n=0}^{\infty} x^n = \frac{1}{1 - x}
\]
\[\sum_{n=-\infty}^{\infty} \beta^{-n} = \sum_{n=1}^{\infty} \beta^{-n} = \sum_{n=0}^{\infty} \beta^{-n} - \beta^0\]

\[= \frac{1}{1-\beta} - 1\]
\[ \sigma_v^2 = \sum_{n=0}^{\infty} \alpha^n + \sum_{n=-\infty}^{-1} \beta^n \]

\[ \alpha = -a(i)e^{-i\omega} \quad \beta = -a(i)e^{i\omega} \]

\[ \text{PSD} = \frac{\sigma_v^2}{1 - a(i)^2} \left\{ \frac{1}{1 + a(i)e^{i\omega}} + \frac{1}{1 + a(i)e^{-i\omega}} - 1 \right\} \]

\[ = \frac{\sigma_v^2}{1 - a(i)^2} \left\{ \frac{2 + a(i) \cdot 2 \alpha e^{i\omega}}{1 + \alpha(i)^2 e^{-i\omega}} - 1 \right\} \]

Inverse Fourier Transform:

\[ r_x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{PSD}(\omega) e^{i\omega t} d\omega \]

\[ = \frac{1}{2\pi} \int_{-\pi}^{\pi} \text{PSD}(\omega) e^{i\omega t} d\omega \quad \text{for} \quad t \geq 0 \]

Title: Nov 3 - 12:33 PM (11 of 15)
Take \( a(n) = 0.8 \) \( \sigma^2_v = 3 \)

\[ d(n) = +0.8 \cdot d(n-1) + V(n) \]

\[ \Rightarrow \]

\[ w(0) = 0.4048 \]

\[ w(1) = 0.2381 \]

\[ d(n) = 0.4048 \cdot x(n) + 0.2381 \cdot x(n-1) \]

\[ d(n) = 0.8 \cdot d(n-1) + \text{noise} \]

\[ x(n) = d(n) + \text{noise} \]
Kalman filter.

Simplified model:

No! \( \bar{d}(n) = a(n) \bar{d}(n-1) + \nu_1(n) \)

Stationary

\[ d(n) = a(n-1) d(n-1) + \nu_1(n) \]

not stationary

Elaboration:

If \( d(n) = a(n) d(n-1) + \nu_1(n) \)

Then \( d(1) = 0.8 d(0) + \)

\[ d(1) = 0.8 d(1) + \]

\[ d(2) = 0.8 d(1) + \]

\[ d(3) = 0.9 d(2) + \]

\[ d(4) = 0.7 d(3) + \]

But \( d(n) = a(n-1) d(n-1) + \nu_1(n) \)

\[ d(1) = 0.8 d(1) + \]

\[ d(2) = 0.9 d(2) + \]

\[ d(3) = 0.7 d(3) + \]
denoted \( d(n) = a(n-1) d(n-1) + v(n) \)

measured \( x(n) = d(n) + \phi(n) v_2(n) \)

Kalman estimator

\[
\hat{d}(n) = A \hat{d}(n-1) + B [x(n) - A \hat{d}(n-1)]
\]

Requirement #1: \( \mathbb{E}[\hat{d}(n)^2] = d(n) \) \( \iff \) will see that 
\[ A = a(n-1) \]

Requirement #2: \( \mathbb{E}[^2] \) is minimal.

\( \Rightarrow \) B is "Kalman gain"
Kalmann filter
\[
\hat{d}(n) = \alpha(n-1) \hat{d}(n-1) + K(n) \left[ x(n) - \alpha(n-1) \hat{d}(n-1) \right]
\]

Bias:
\[
\mathbb{E} [ \hat{d}(n) ] = \mathbb{E} [ \hat{d}(n-1) ] + K(n) \mathbb{E} [ x(n) ]
\]

Assume \( \mathbb{E} [ \hat{d}(n-1) ] = d(n-1) \)

\then:
\[
\mathbb{E} [ \hat{d}(n) ] = \mathbb{E} [ \hat{d}(n-1) ] + K(n) d(n) - K(n) \alpha(n-1) d(n-1)
\]
\[= K(n) d(n) + \left[ 1 - K(n) \right] \alpha(n-1) d(n-1)
\]
\[d(n) - v(n)\]