Kalman:

You have an estimate of the previous value $\hat{d}_{old}$, unbiased, estimated error $\mathcal{E}[e_{old}^2]$.

You have a model of how the new value evolves:

$$d_{new} = a\, d_{old} + v_1 \text{ (noise)}$$

You have a noisy measurement of d_{new}:

$$x_{new} = d_{new} + v_2 \text{ noise}$$

Goal: estimate d_{new} :

① Use model :

$$\hat{d}_{new} = a\,\hat{d}_{old}$$

② Use measurement result :

$$\hat{d}_{new} = x_{new}$$

Kalman: take a weighted average

$$\hat{d}_{new} = K\,x_{new} + (1-K)\,a\,\hat{d}_{old}$$

$$= a\hat{d}_{old} + K\left[x_{new} - a\hat{d}_{old}\right]$$

If measurement is perfect, $\sigma_{v_2} = 0$, you would use

$$\hat{d}_{new} = x_{new} \qquad (K=1)$$

If measurement is lousy $\sigma_{v_2} = \infty$, ignore it and

$$\hat{d}_{new} = a\,\hat{d}_{old} .$$

In general, the <u>optimal</u> K will depend on σ_{v_1} <u>and</u> σ_{v_2} .

If you need to estimate the next d, you have

(1) the preceding estimate $\hat{d}$ (you just computed it)

(2) the model $d_{new, improved} = b\, d_{new} + \overset{\text{noise}}{V_1}$

(3) new measurement $X_{new, improved} = d_{new, improved} + V_2$

(4) you need $E[e_{new}^2]$

Kalman algorithm gives you the K and the $E[e^2]$

More complicated:

$$d_{\text{today}} = .5\, d_{\text{yesterday}} + .4\, d_{\text{day-before-yesterday}} + \overset{\text{noise}}{v_1}$$

State-space trick.

$$d(n) = .5\, d(n-1) + .4\, d(n-2) + v_1(n)$$

matrix formulation

$$\begin{bmatrix} d(n) \\ d(n-1) \\ d(n-2) \end{bmatrix} = \begin{bmatrix} .5 & .4 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} d(n-1) \\ d(n-2) \\ d(n-3) \end{bmatrix} + \begin{bmatrix} v_1 \\ 0 \\ 0 \end{bmatrix}$$

✓ $\vec{d}(n) = A\, \vec{d}(n-1) + \vec{v}$ ←

looks ~~best~~ like $d(n) = a(n-1)\, d(n-1) + v_1(n)$

Measurement

$$x(n) = d(n) + v_2(n)$$

$$x(n) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} d(n) \\ d(n-1) \\ d(n-2) \end{bmatrix} + v_2(n)$$

$$\vec{d}(n)$$

$$x = \vec{c}\,\vec{d} + v_2$$

looks like $x(n) = \underset{\lambda}{c}\, d(n) + v_2(n)$

The matrix Kalman problem has

~~an~~ a model : $\vec{d}(n) = \overleftrightarrow{A}(n-1)\vec{d}(n-1) + \vec{v_1}(n)$

measurement : $x(n) = \vec{c}(n)^{\text{Transpose}}\vec{d}(n) + v_2(n)$

Start with an unbiased estimate $\hat{\vec{d}}(n-1)$

and its expected error : $\left(\mathcal{E}\{e_{n-1}^2\}\right) \leftarrow ?$

$$\begin{bmatrix} e \end{bmatrix} = \begin{bmatrix} \hat{d} \end{bmatrix} - \begin{bmatrix} d \end{bmatrix}$$

$$e^2 = \begin{bmatrix} e \end{bmatrix} \begin{bmatrix} e \end{bmatrix}$$ No, not defined

$$e^2 = \begin{bmatrix} e^T \end{bmatrix} \begin{bmatrix} e \end{bmatrix}$$ This is defined, a sum of squares of errors, a scalar

Use this one $\Longrightarrow$ $$e^2 = \begin{bmatrix} e \end{bmatrix} \begin{bmatrix} e^T \end{bmatrix}$$

$$= \begin{bmatrix} a \\ b \\ c \end{bmatrix} \begin{bmatrix} a & b & c \end{bmatrix} = \begin{bmatrix} a^2 & ab & ac \\ ba & b^2 & bc \\ ca & cb & c^2 \end{bmatrix}$$

So use $\mathcal{E}[e e^T] = \mathcal{E}\left\{ \begin{bmatrix} e(n) \\ e(n-1) \\ e(n-2) \end{bmatrix} \begin{bmatrix} e(n) & e(n-1) & e(n-2) \end{bmatrix} \right\}$

$$= \begin{bmatrix} \mathcal{E}[e(n)^2] & \mathcal{E}[e(n)e(n-1)] & \mathcal{E}[e(n)e(n-2)] \\ & \mathcal{E}[e(n-1)^2] & \vdots \\ \ddots & \cdots & \mathcal{E}[e(n-2)^2] \end{bmatrix}$$

What does $\hat{d}(n)$ look like?

scalar case

$$\hat{d}(n) = a(n-1)\,\hat{d}(n-1) + K\left[x(n) - a(n-1)\cdot\hat{d}(n-1)\right]$$

matrix case

$$\hat{d}(n) = A(n-1)\,\hat{d}(n-1) + K(n)\left[x(n) - C(n)\,A(n-1)\,\hat{d}(n-1)\right]$$

everything is a matrix