Basic theorems:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$x^T A x$ $\quad$ A is $n \times n$ symmetric matrix:

$$\nabla (x^T A x) = 2 A x$$

$x^T a$ $\quad$ a is $n \times 1$ column vector

$$\nabla (x^T a) = a$$

Bilinear

$ax^2 + by^2 + cz^2 + dxy + eyz + fxz$

$$[x\ y\ z]\begin{bmatrix} A & B & C \\ D & E & F \\ G & H & I \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = [x\ y\ z]\begin{bmatrix} a & d/2 & f/2 \\ d/2 & b & e/2 \\ f/2 & e/2 & c \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$[x\ y\ z]\begin{bmatrix} Ax + By + Cz \\ Dx + Ey + Fz \\ Gx + Hy + Iz \end{bmatrix}$$

$$= Ax^2 + Bxy + Cxz + Dyx + Ey^2 + Fyz + Gxz + Hzy + Iz^2$$

Note that matrix element in i^{th} row, j^{th} column ends up multiplying x_i and x_j

$(B + D \equiv d$; $B = 0,\ D = d$ or
$B = \frac{1}{3}d,\ D = \frac{2}{3}d$ or

Best $\longrightarrow$ $B = \frac{1}{2}d = D$. $)$

$$[x_1\ x_2\ x_3]\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{3} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$A_{21} = A_{12}$
$A_{31} = A_{13}$
$A_{32} = A_{23}$

$$= A_{11}x_1^2 + A_{22}x_2^2 + A_{33}x_3^2$$
$$+ 2A_{12}x_1x_2 + 2A_{23}x_2x_3 + 2A_{13}x_1x_3$$

From the red box on page 2 we see

$$\begin{bmatrix} x_1 & x_2 & x_n \end{bmatrix} \begin{bmatrix} A_{11} & & & \\ & \cdots & A_{ij} & \cdots \\ & & & \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

equals

$$\boxed{\sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} x_i x_j}.$$

Summary

$$a x_1^2 + b x_1 x_2 + c x_2^2 + d x_1 x_3 + e x_3^2 + \cdots$$

$$\parallel\!\!\!\mid$$

$$[x_1, x_2 - x_n] \begin{bmatrix} \text{symmetric} \\ \text{matrix} \\ A_{ij} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$\parallel$$

$$\sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} x_i x_j .$$

Look $ax + by + cz$ <u>linear</u> form

$$= \begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$x^T a = a^T x$

$$= \sum_{i=1}^{n} a_i x_i \ .$$

$$\frac{\partial \text{ linear form}}{\partial x_2} = b \equiv a_2$$

$$\frac{\partial \text{ ———}}{\partial x_1} = a_1$$

$$\frac{\partial \text{ ——}}{\partial x_3} = a_3$$

$$\begin{bmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} \\ \vdots \\ \frac{\partial}{\partial x_n} \end{bmatrix} (a^T x) = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} a \end{bmatrix}$$

"gradient" $\equiv \begin{bmatrix} \nabla \end{bmatrix}$

Bilinear

$ax^2 + by^2 + cz^2 + dxy + eyz + fxz$

$$\begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} A & B & C \\ D & E & F \\ G & H & I \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} a & d/2 & f/2 \\ d/2 & b & e/2 \\ f/2 & e/2 & c \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} Ax + By + Cz \\ Dx + Ey + Fz \\ Gx + Hy + Iz \end{bmatrix}$$

$= Ax^2 + Bxy + Cxz + Dyx + Ey^2 + Fyz$
$+ Gxz + Hzy + Iz^2$

Note that matrix element in i^{th} row, j^{th} column ends up multiplying x_i and x_j

$(B + D \equiv d;$ $B = 0, D = d$ or
$B = \frac{1}{3}d, D = \frac{2}{3}d$ or

Best $\longrightarrow$ $B = \frac{1}{2}d = D$.)

Take $\frac{\partial}{\partial x_2} = \frac{\partial}{\partial y}$ (bilinear form):

$\frac{\partial}{\partial y}(\text{scalar form}) = 2by + dx + ez$

$\equiv 2A_{22}x_2 + 2A_{21}x_1 + 2A_{23}x_3$

$= 2 \begin{bmatrix} A_{21} & A_{22} & A_{23} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$\frac{\partial}{\partial x} = 2 \begin{bmatrix} A_{11} & A_{12} & A_{13} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$\frac{\partial}{\partial z} = 2 \begin{bmatrix} A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$$\nabla(x^T A x) = \begin{bmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{bmatrix} x^T A x = 2 \begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} \begin{bmatrix} \\ x \\ \\ \end{bmatrix}$$

So:

$$\nabla x^T A x = 2Ax$$ (A symmetric)

$$\nabla x^T a = \nabla a^T x = a$$

Wiener Filter in general.

Measure $x(n)$ } stationary

Wish to estimate $d(n)$ }

Assume: Know $r_x(n)$ & $r_{dx}(n)$.

Estimator:
$$\hat{d}(n) = \sum_{K=K_1}^{K_2} x(n-K)\, w(K)$$

application

$$\hat{d}(n) = x(n)\, w(0)$$ denoiser

$$\hat{d}(n) = x(n)\, w(0) + x(n-1)\, w(1) + x(n-2)\, w(2)$$

3-term denoiser

$$\hat{d}(n) = x(n)\,w(0) + x(n-1)\, w(1)$$

$$x(n+1)\, w(-1)$$

smoother

~~$d(n) = x(n+1)$~~

$d(n) = x(n+1)$, $\hat{d}(n) = x(n)\, w(0) + x(n-1)\, w(1) + x(n-2)\, w(2)$

predictor

In general

$$\hat{d}(n) = \sum_{K} x(n-K)\, w(K)$$

$$= \left[x(n) \;\; x(n-1) \cdots \right] \left[\begin{matrix} w(0) \\ w(1) \\ \vdots \\ w(\cdot\cdot) \end{matrix} \right]$$

$$= x^T w$$

error $\hat{d}(n) - d(n) = x^T w - d$

squared error $= [x^T w - d][x^T w - d]$

$$(\text{error})^2 = [x^T w - d][x^T w - d]$$

$$= [w^T x - d][x^T w - d]$$

$$= w^T x x^T w - d x^T w - w^T x d + d^2$$

$$E\{e^2\} = w^T E[x x^T] w - 2 E[d x^T] w + d^2$$

$$x x^T = \begin{bmatrix} x(n) \\ x(n-1) \\ x(n-2) \\ \vdots \end{bmatrix} \begin{bmatrix} x(n) & \cdots & x(n-) \end{bmatrix}$$

$$= \begin{bmatrix} x(n)^2 & x(n)x(n-1) & x(n)x(n-2) & x(n)x(n-k) \\ x(n-1)x(n) & x(n-1)x(n-1) & x(n-1)x(n-2) & \\ & & & \end{bmatrix}$$

$$E[x x^T] = \begin{bmatrix} r_x(0) & r_x(1) & \\ r_x(1) & \ddots & \\ & & \ddots \end{bmatrix}$$ symmetric matrix autocorrelation

$$d x^T = d(n)\left[x(n) \quad x(n-1) \quad \cdots \quad x(n-K) \right]$$

$$\mathcal{E}\left[d x^T \right] = \left[r_{dx}(0) \quad r_{dx}(1) \quad \cdots \quad r_{dx}(n) \right]$$

vector of cross-correlations.

$$\mathcal{E}\{e^2\} = w^T \, \mathcal{E}[x x^T] w - 2\, \mathcal{E}[d x^T]\, w + d^2$$

$$\nabla_w \mathcal{E}[e^2] = 0 = 2\, \underbrace{\mathcal{E}[x x^T]}_{r_d} w - 2\, \underbrace{\mathcal{E}[d x]}_{r_{dx}} = 0$$

to minimize

Wiener-Hopf

$$r_d w = r_{dx}$$

$$w = r_d^{-1} r_{dx}$$