Wiener Filter example.

Random process $d(n)$

Measurement of $d(n)$ is corrupted with noise

$$x(n) = d(n) + v(n)$$

Everything is stationary.

~~Wi~~ Wiener problem: what's the best way to process $\{x(n)\}$ to estimate $d(n)$?

Strategy of Wiener filter:

estimate $d(n)$ by

$$\hat{d}(n) = w(-1)x(n+1) + w(0)x(n) + w(1)x(n-1) + w(2)x(n-2)$$

Scenarios:

estimate $d(n)$ from ~~w(n)~~ $x(n)$

(de-noising).

$w(0) \neq 0$, $w(-1), w(1), w(2), \ldots = 0$

estimate $d(n)$ from $x(n)$ and previous measurements:

$$\hat{d}(n) = w(0)\,x(n) + w(1)\,x(n-1) + w(2)\,x(n-2).$$

(denoising)

estimate $d(n)$ from $x(n)$, $x(n-1)$, $x(n+1)$.

(smoothing)

$w(0), w(1), w(-1) \neq 0$

others $= 0$.

estimate $d(n)$ from $x(n-1), x(n-2), x(n-3)$

(prediction)

$w(1), w(2), w(3) \neq 0$

others $= 0$.

~~Today~~ Today's problem: $x(n) = d(n) + v(n)$

estimate $d(n)$ from $x(n)$ & $x(n-1)$

$$\hat{d}(n) = w(0)\,x(n) + w(1)\,x(n-1)$$

Least Mean Square ~~Est~~ Estimator.

LMS Estimator.

$$\text{Error} = \hat{d}(n) - d(n) = w(0)\,x(n) + w(1)\,x(n-1) - d(n)$$

$$E\left[|\text{Error}|^2\right] = E\left[\hat{d}(n)^2\right] + E\left[d(n)^2\right] - 2E\left[\hat{d}(n)\,d(n)\right]$$

$$= E\left\{ w(0)^2 x(n)^2 + w(1)^2 x(n-1)^2 + 2w(0)\,w(1)\,x(n)\,x(n-1) \right\}$$

$$+ \cancel{r_D(0)^2} - 2E\left\{ \left(w(0)\,x(n) + w(1)\,x(n-1)\right) d(n) \right\}$$

$$E\left[|Error|^2\right] = E\left[\hat{d}(n)^2\right] + E\left[d(n)^2\right] - 2E\left[\hat{d}(n)\, d(n)\right]$$

$$= E\left\{ w(0)^2 x(n)^2 + w(1)^2 x(n-1)^2 + 2w(0)w(1)\,x(n)\,x(n-1) \right\}$$

$$+ r_D(0) - 2E\left\{ \left(w(0)x(n) + w(1)x(n-1)\right) d(n) \right\}$$

$$= w(0)^2 r_x(0) + w(1)^2 r_x(0) + 2w(0)w(1)\, r_x(1)$$

$$+ r_d(0) - 2w(0)\, r_{xD}(0) - 2w(1)\, r_{xD}(-1)$$

$$r_{xD}(m) = E\left\{ x(n+m)\, D(n) \right\}$$

$$E\left[x(l)\,x(m)\right] = r_x(l-m) = r_x(m-l)$$

$$E\left[x(l)\, d(m)\right] \triangleq r_{xD}(l-m)$$

$$E\left[x(m)\, d(l)\right] = r_{xD}(m-l)$$

$$E\left[x(n-1)\, d(n)\right] = r_{xD}(-1)$$

$$\shortparallel$$

$$E\left[d(n)\, x(n-1)\right] = r_{Dx}(+1)$$

To match my YulWalWienKal.doc notes,

$$MSE = w(0)^2\, r_x(0) + w(1)^2\, r_x(0) + 2\,w(0)\,w(1)\, r_x(1)$$
$$\qquad - 2\,w(0)\, r_{dx}(0) - 2\,w(1)\, r_{dx}(1) + r_d(0)$$

To get "LEAST" MS error, minimize:

$$0 = \frac{\partial\, MSE}{\partial\, w(0)} = 2\,w(0)\,r_x(0) + 2\,w(1)\,r_x(1) - 2\,r_{dx}(0)$$

$$0 = \frac{\partial\, MSE}{\partial\, w(1)} = 2\,w(1)\,r_x(0) + 2\,w(0)\,r_x(1) - 2\,r_{dx}(1)$$

Wiener-Hopf Equations

$$\begin{bmatrix} r_x(0) & r_x(1) \\ r_x(1) & r_x(0) \end{bmatrix} \begin{bmatrix} w(0) \\ w(1) \end{bmatrix} = \begin{bmatrix} r_{dx}(0) \\ r_{dx}(1) \end{bmatrix}$$

How do get ~~get~~ $r_x(-)$, $r_{dx}(-)$?

Use model:

$$x(n) = d(n) + v(n)$$

$$r_{dx}(k) = E\{ d(n) \underbrace{x(n-k)}_{d(n-k)+v(n-k)} \} =$$

$$= \underbrace{E\{d(n)\,d(n-k)\}}_{= r_d(k)} + \underbrace{E\{d(n)\,v(n-k)\}}_{= E\{d(n)\}\,E\{v(n-k)\} = 0}$$

$$r_{dx}(k) = r_d(k)$$

for this model

not always!

$$x(n) = d(n) + v(k)$$

$$r_x(k) = E\left\{ x(n)\, x(n-k) \right\}$$

$$= E\left\{ \big(d(n) + v(n)\big)\big(d(n-k) + v(n-k)\big) \right\}$$

$$= r_d(k) + 0 + 0 + \begin{cases} 0 & \text{if } k \neq 0 \\ \sigma_v^2 & \text{if } k = 0 \end{cases}$$

$$r_x(k) = r_d(k) + \sigma_v^2\, \delta_{k0}$$

$$\begin{bmatrix} r_x(0) & r_x(1) \\ r_x(1) & r_x(0) \end{bmatrix} \begin{bmatrix} w(0) \\ w(1) \end{bmatrix} = \begin{bmatrix} r_{dx}(0) \\ r_{dx}(1) \end{bmatrix}$$

$$\begin{bmatrix} r_d(0) + \sigma_v^2 & r_d(1) \\ r_d(1) & r_d(0) + \sigma_v^2 \end{bmatrix} \begin{bmatrix} w(0) \\ w(1) \end{bmatrix} = \begin{bmatrix} r_d(0) \\ r_d(1) \end{bmatrix}$$