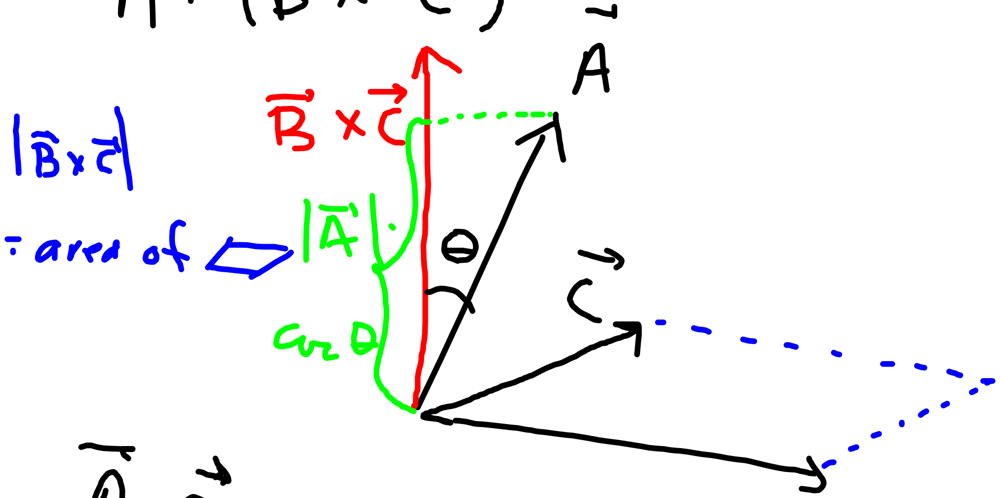
§1.13 Triple Scalar Product

$$\vec{A} \cdot (\vec{B} \times \vec{C})$$

Triple Cross Product

$$\vec{A} \times (\vec{B} \times \vec{C})$$

$\vec{A}\cdot(\vec{B}\times\vec{C})$



$\vec{A}\cdot\vec{B}\times\vec{C} = |\vec{A}|\;\underbrace{|\vec{B}\times\vec{C}|}_{\text{area of ▱}}\cos\theta$ $\quad$ ($|\vec{A}|\cos\theta = \text{height}$)

= area of base × height

$\vec{A}\cdot(\vec{B}\times\vec{C}) = \text{VOLUME}$

Analytic form for $\vec{A}\cdot\vec{B}\times\vec{C}$

$$\vec{A}\cdot\vec{B}\times\vec{C} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} \cdot (A_1\vec{i} + A_2\vec{j} + A_3\vec{k})$$

$$\vec{A}\cdot\vec{B}\times\vec{C} = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}$$

Consequences

1. A determinant IS a volume.
2. Any " identity becomes a triple scalar product identity.

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}$$

$$\vec{A}\cdot(\vec{C}\times\vec{B}) = -\vec{A}\cdot(\vec{B}\times\vec{C})$$

$$\vec{C}\cdot(\vec{A}\times\vec{B}) = \vec{A}\cdot(\vec{B}\times\vec{C})$$

$$\vec{A}\times\vec{B}\cdot\vec{C} = \vec{C}\cdot(\vec{A}\times\vec{B}) = \vec{A}\cdot(\vec{B}\times\vec{C})$$

Application.

Given 4 vectors: $\vec{A}, \vec{B}, \vec{C}, \vec{D}$

Task: write $\vec{D} = x\vec{A} + y\vec{B} + z\vec{C}$ $\quad x, y, z = ?$

Case 1. $\vec{A} = \vec{i} \quad \vec{B} = \vec{j} \quad \vec{C} = \vec{k}$

$$\vec{D} = D_1\vec{i} + D_2\vec{j} + D_3\vec{k} \quad \text{trivial}$$

Case 2. $\vec{A} = 2\vec{i} \quad \vec{B} = 3\vec{j} \quad \vec{C} = 4\vec{k}$

$$\vec{D} = \frac{D_1}{2}\vec{A} + \cancel{\frac{\vec{D}\cdot\vec{B}}{3\vec{B}\cdot\vec{B}}}\frac{D_2}{3}\vec{B} + \frac{D_3}{4}\vec{C}$$

General Case $\vec{D} = x\vec{A} + y\vec{B} + z\vec{C}$ $\quad \vec{A}, \vec{B}, \vec{C}$ not $\perp$, not unit.

Suppose you know a vector that was $\perp B$ and $\perp C$: call it $\vec{P}$

$$\vec{D}\cdot\vec{P} = x\vec{A}\cdot\vec{P} \Rightarrow x = \frac{\vec{D}\cdot\vec{P}}{\vec{A}\cdot\vec{P}}$$

Final Answer: choose $\vec{P} = \vec{B}\times\vec{C}$

$$\vec{D} = x\vec{A} + y\vec{B} + z\vec{C}$$

$$\vec{D}\cdot(\vec{B}\times\vec{C}) = x\,\vec{A}\cdot(\vec{B}\times\vec{C}) + y\cdot 0 + z\cdot 0$$

$$x = \cancel{\frac{A\cdot B\times C}{D\cdot B\times C}} \quad \frac{\vec{D}\cdot\vec{B}\times C}{\vec{A}\cdot\vec{B}\times\vec{C}}$$

(Correction added later)

$$\vec{D} = x\vec{A} + y\vec{B} + z\vec{C}$$

$$D_1\vec{i} + D_2\vec{j} + D_3\vec{k} = xA_1\vec{i} + xA_2\vec{j} + xA_3\vec{k} + yB_1\vec{i} + \dots + zC_3\vec{k}$$

$$D_1 = xA_1 + yB_1 + zC_1$$

$$D_2 = xA_2 + yB_2 + zC_2$$

$$D_3 = xA_3 + yB_3 + zC_3$$

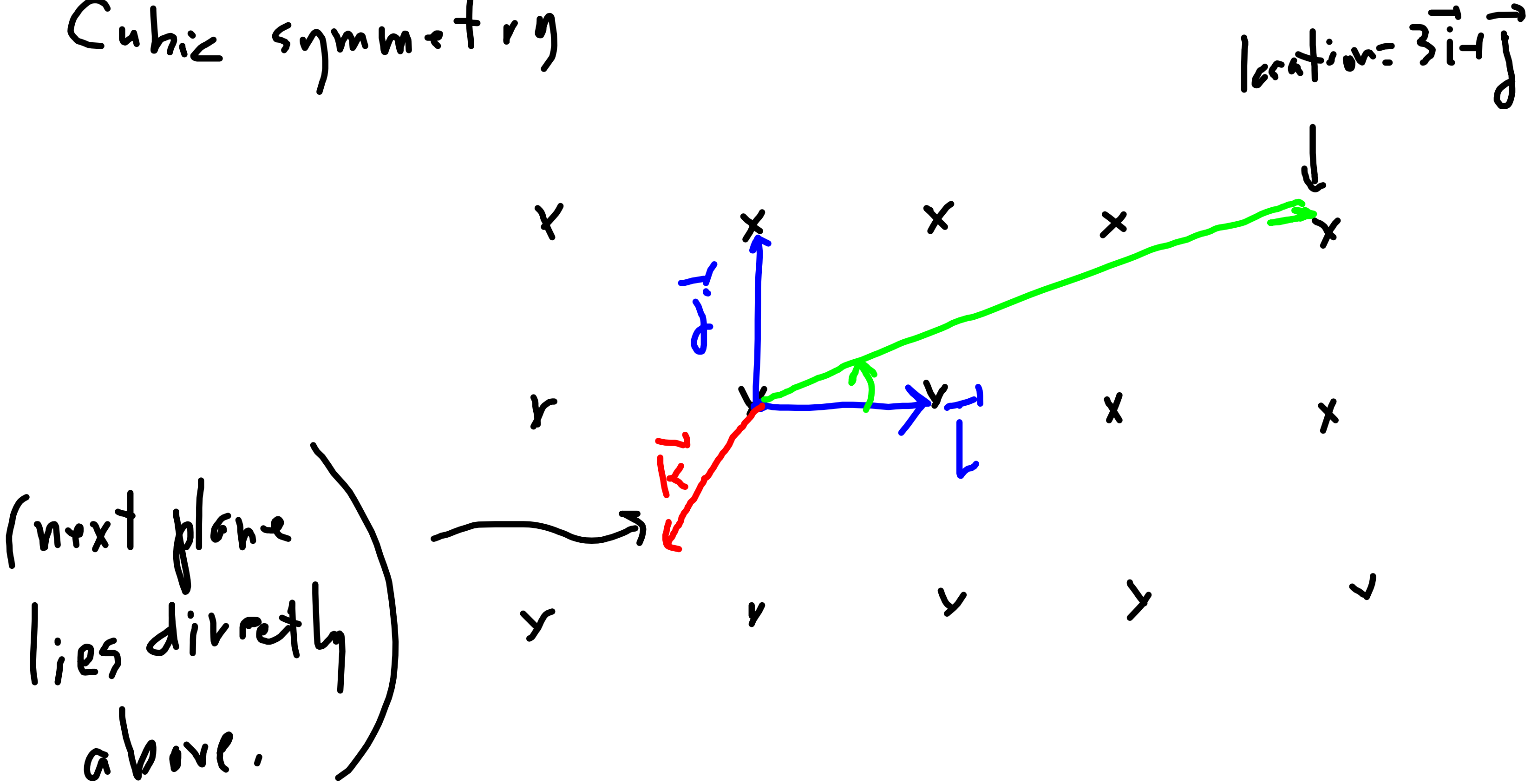
$$\begin{bmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} D_1 \\ D_2 \\ D_3 \end{bmatrix}$$

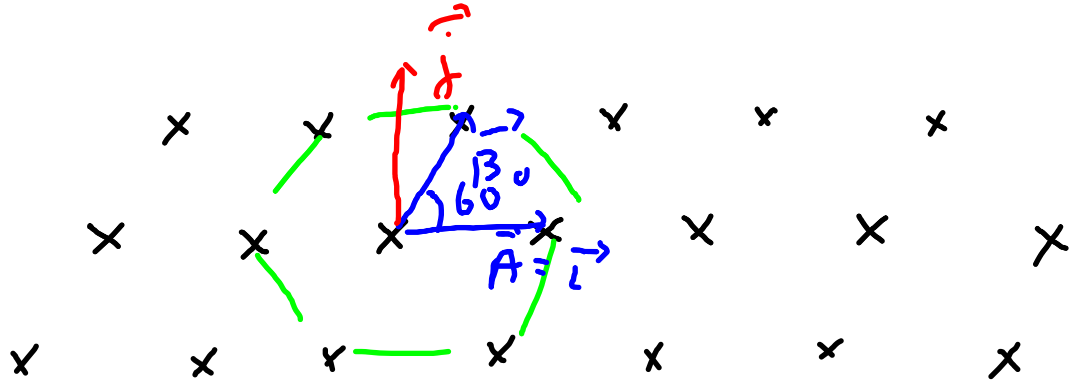
Cramer's Rule

$$x = \frac{\begin{vmatrix} D_1 & B_1 & C_1 \\ D_2 & B_2 & C_2 \\ D_3 & B_3 & C_3 \end{vmatrix}}{\begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix}} = \frac{\begin{vmatrix} D_1 & D_2 & D_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}}{\begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}} = \frac{\vec{D}\cdot\vec{B}\times\vec{C}}{\vec{A}\cdot\vec{B}\times\vec{C}}$$

Application: Chr
Xtalography

Cubic symmetry





next plane lies directly above.

Suppose a site is at $4\vec{i} + 7\vec{j} + 0\vec{k}$

How do you ~~go~~ get there moving along atoms in the chrystal?

crystal

$$4\vec{i} + 7\vec{j} = \underline{?}\,\vec{A} + \underline{?}\,\vec{B}$$

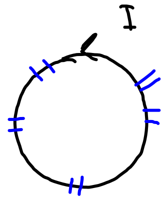
Triple Vector Product

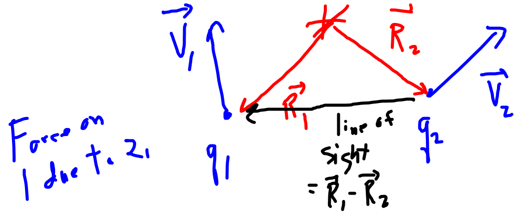
$$\vec{A} \times (\vec{B} \times \vec{C})$$

Visually, (?)

Analytically $\vec{A} \times (\vec{B} \times \vec{C}) = \vec{A} \cdot \vec{C} \, \vec{B} - \vec{A} \cdot \vec{B} \, \vec{C}$

Ampere's Law.





Force $\propto \dfrac{1}{\text{distance}^2}$

Force on 1 $\propto$ Magnetic field at 1 due to 2.

$\propto \dfrac{1}{\text{distance}^2}$, q_2, $|\vec V_2|$

$\perp$ to line of sight

$\perp$ to $\vec V_2$

magnetic field $\quad \vec B = K q_2 \dfrac{\vec V_2 \times (\vec R_1 - \vec R_2)}{|\vec R_1 - \vec R_2|^3}$

$$\vec F_{2\,on\,1} = \frac{K q_1 q_2 \, \vec V_1 \times \left[\vec V_2 \times (\vec R_1 - \vec R_2)\right]}{|\vec R_1 - \vec R_2|^3}$$

$$\vec{F}_{2\,on\,1} = \frac{K\, q_1 q_2\, \vec{V}_1 \times \left[\vec{V}_2 \times (\vec{R}_1 - \vec{R}_2)\right]}{|\vec{R}_1 - \vec{R}_2|^3}$$