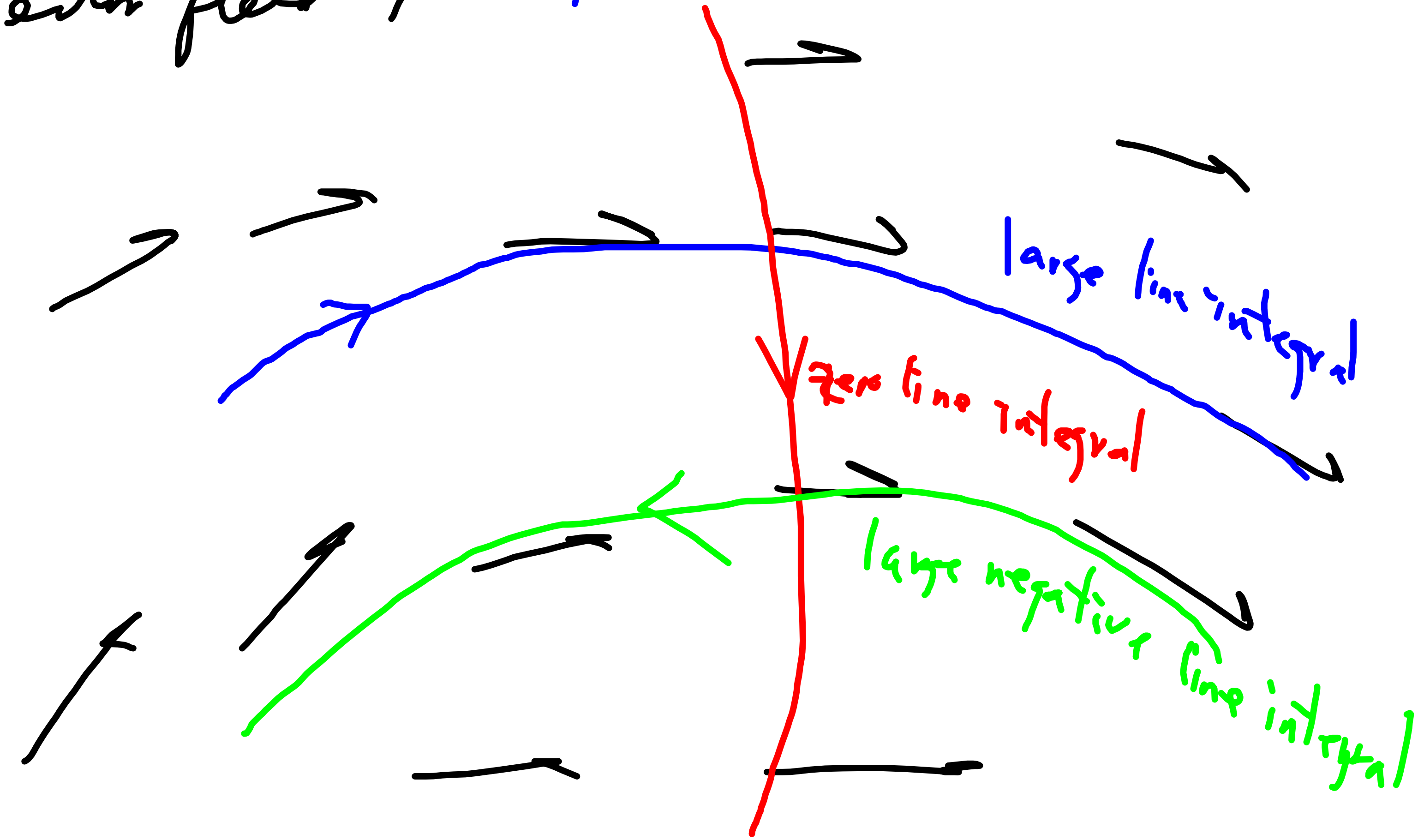
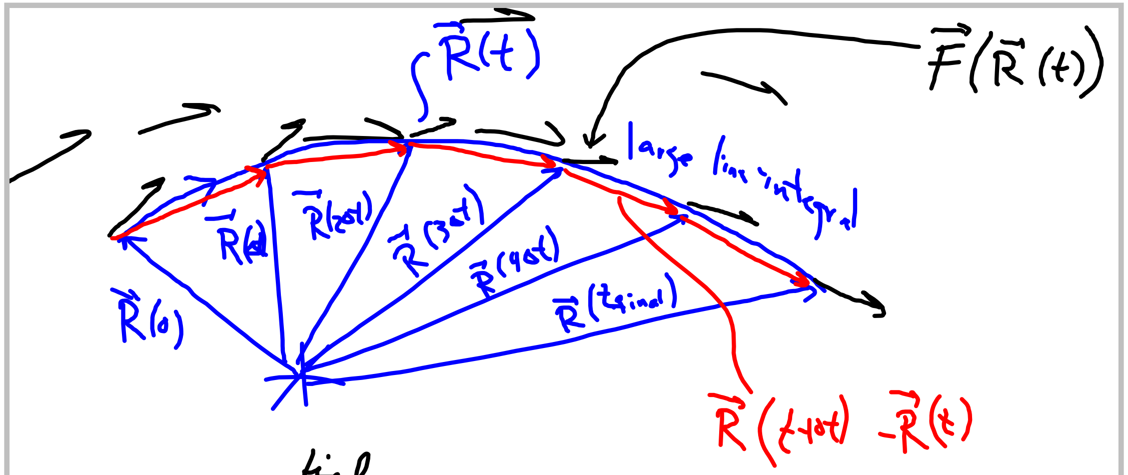
§4.1 Line Integral

Vector field $\vec{F}$; curve





$$\lim_{\Delta t \to 0} \sum_{start}^{final} \vec{F}(\vec{R}(t)) \cdot \left\{ \vec{R}(t+\Delta t) - \vec{R}(t) \right\}$$

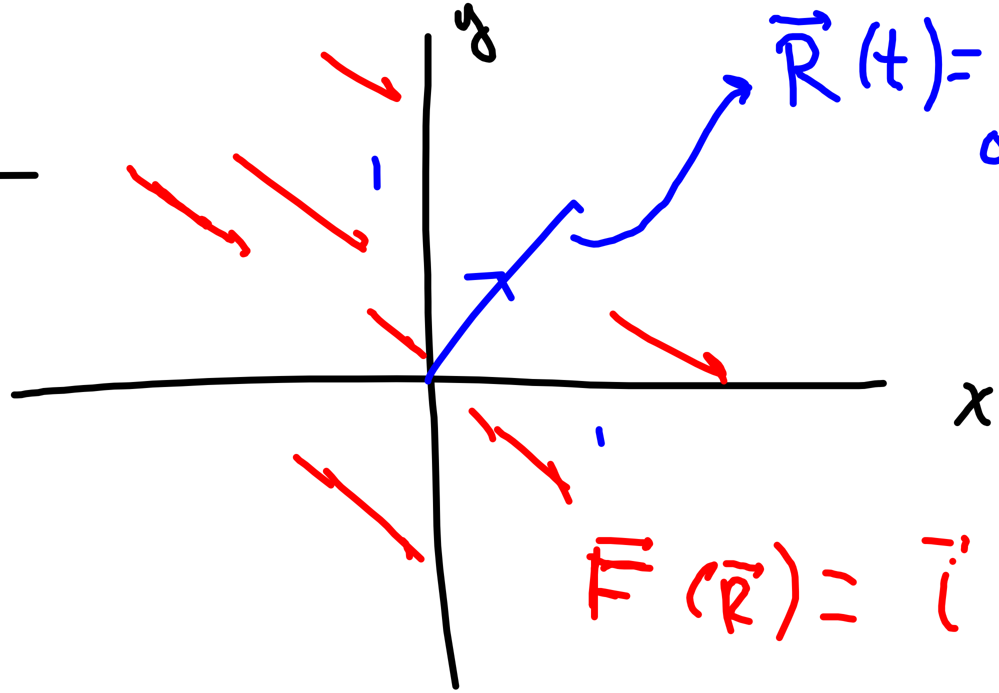
good for computer

$$= \vec{F}(\vec{R}(t)) \cdot \frac{\Delta R(t)}{\Delta t} \Delta t$$

$$\text{“} \int_C \vec{F} \cdot d\vec{R} \text{”} = \int_{t_{start}}^{t_{final}} \vec{F}(\vec{R}(t)) \cdot \frac{d\vec{R}(t)}{dt} \, dt$$

analytic

Example



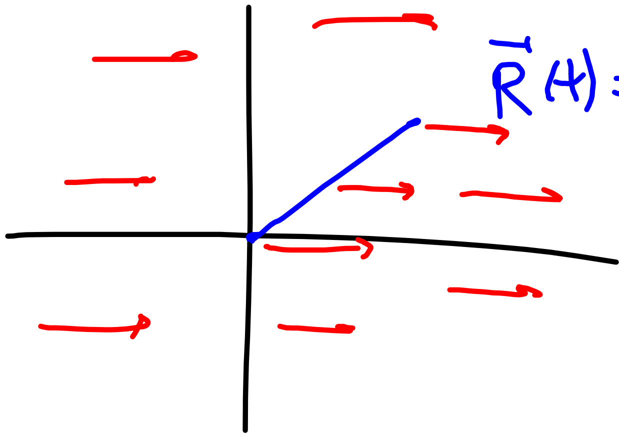
$$\vec{R}(t) = \vec{0} + t(\vec{i} + \vec{j}) \quad 0 \le t \le 1$$

$$(z = 0)$$

$$\vec{F}(\vec{R}) = \vec{i} - \vec{j}$$

$$\int_C \vec{F} \cdot d\vec{R} \text{ obviously } = 0 = \int_{t=0}^{1} \underbrace{(\vec{i} - \vec{j}) \cdot (\vec{i} + \vec{j})}_{0}\, dt$$

$\vec{F} = \vec{i}$



$\vec{R}(t) = \vec{0} + t(\vec{i} + \vec{j})$

$\int_C \vec{F} \cdot d\vec{R}$ obviously positive, obviously less than

$|\vec{F}|_{\text{highest}} \, |\text{length of curve}|$

$= 1 \sqrt{2}$

$$\int \vec{F} \cdot d\vec{R} = \int_{t=0}^{1} \underbrace{\vec{i} \cdot (\vec{i} + \vec{j})}_{1} \, dt = 1$$

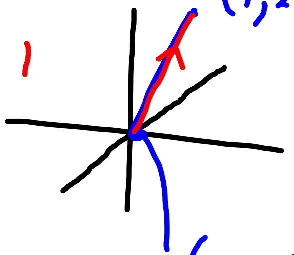
Example (p. 188)

$$\vec{F}(\vec{R}) = x^2\,\vec{i} + y\,\vec{j} + (xz - y)\,\vec{k}$$

$\vec{R}(t) = \vec{O} + t[\vec{i} + 2\vec{j} + 4\vec{k}]$

$0 \le t \le 1$

(1, 2, 4)



(0, 0, 0)

$$\int \vec{F} \cdot d\vec{R} = \int_{t=0}^{1} [x^2\vec{i} + y\vec{j} + (xz - y)\vec{k}] \cdot [\vec{i} + 2\vec{j} + 4\vec{k}]\,dt$$

$$= \int_0^1 [t^2\vec{i} + 2t\vec{j} + (4t^2 - 2t)\vec{k}] \cdot [\vec{i} + 2\vec{j} + 4\vec{k}]\,dt$$

$$\vec{R}(t) = x(t)\,\vec{i} + y(t)\,\vec{j} + z(t)\,\vec{k}$$
$$= t\,\vec{i} + 2t\,\vec{j} + 4t\,\vec{k}$$

$$= \int_0^1 \left(t^2 + 4t + 4[4t^2 - 2t]\right) dt$$

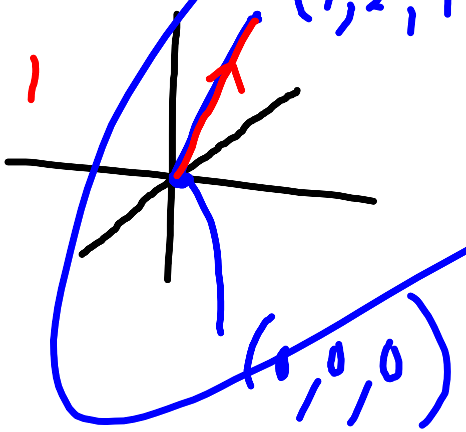
$$= 11/3$$

Example (p. 188)

$$\vec{F}(\vec{R}) = x^2\,\vec{i} + y\,\vec{j} + (xz - y)\,\vec{k}$$

$\vec{R}(t) = \vec{0} + t[\vec{i} + 2\vec{j} + 4\vec{k}]$

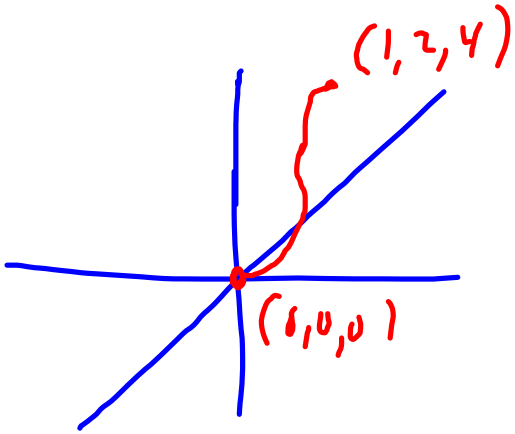
$0 \le t \le 1$



$$\int \vec{F} \cdot d\vec{R}$$

$\vec{i}$ 0

$$\vec{F} = x^2\,\vec{i} + y\,\vec{j} + (xz - y)\,\vec{k}$$



curve $x = t^2$ $\quad 0 \le t \le 1$

$y = 2t$

$z = 4t^3$

$$\vec{R}(t) = t^2\,\vec{i} + 2t\,\vec{j} + 4t^3\,\vec{k}$$

$$\int \vec{F}\cdot d\vec{R} = \int \left\{ \left[t^4\,\vec{i} + 2t\,\vec{j} + \left[4t^5 - 2t\right]\vec{k} \right\} \cdot \left[2t\,\vec{i} + 2\,\vec{j} + 12t^2\,\vec{k} \right] dt$$

$$= 7/3$$

Other nomenclature:

line integral along curve

contour integral

integral of the tangential component of $\vec{F}$ along C.

$$\int_C \vec{F} \cdot d\vec{R}$$

$$\int_C F_{tan}\, \underbrace{ds}_{\text{arc length}}$$

$$\int \left(F_1\, dx + F_2\, dy + F_3\, dz \right)$$

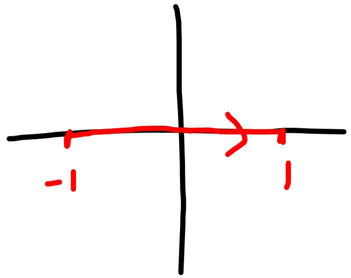
All mean the same thing

Example $\vec{F} = x\vec{i} + x^2\vec{j}$ start $(-1,0,0)$

end $(+1,0,0)$

in $z=0$ plane.

Part one along the x-axis



$$\vec{R}(t) = -\vec{i} + t(2\vec{i}) \qquad 0 \le t \le 1$$

$$\text{or} \quad = -\vec{i} + t(\vec{i}) \qquad 0 \le t \le 2$$

$$\text{or} \quad = \vec{0} + t\vec{i} \qquad -1 \le t \le 1$$

$$\int \vec{F}\cdot d\vec{R} = \int_{t=0}^{1} \left[(2t-1)\vec{i} + (2t-1)^2\vec{j}\right]\cdot\left[2\vec{i}\right] dt$$

(use first one)

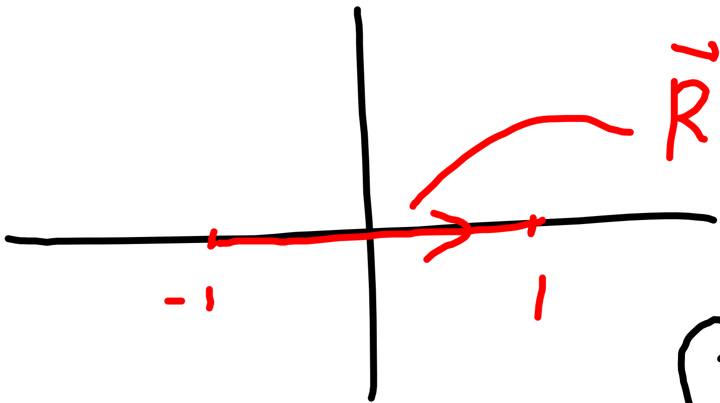
$$= \int_0^1 (2t-1)\cdot 2\, dt = 0$$

<u>Example</u> $\vec{F} = x\vec{i} + x^2\vec{j}$ start $(-1,0,0)$

end $(+1,0,0)$

in $z=0$ plane.

along the x-axis (again, but smarter)



$\vec{R} = x\vec{i}$, $-1 \le x \le 1$

$$\int \vec{F}\cdot d\vec{R} = \int_{x=-1}^{x=1} \left(x\vec{i} + x^2\vec{j}\right)\cdot\vec{i}\,dx$$

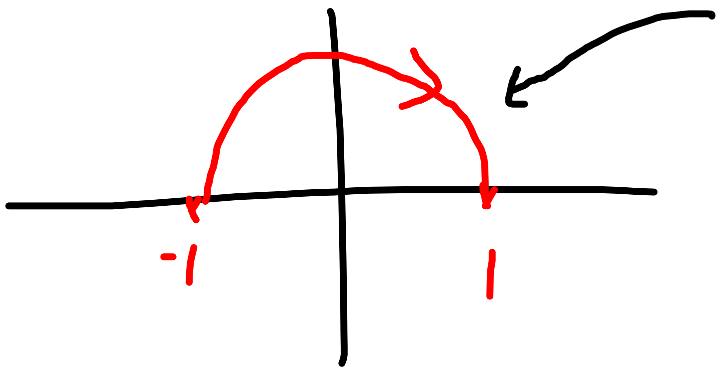
$$= \int_{-1}^{1} x\,dx = 0$$

Example $\vec{F} = x\,\vec{i} + x^2\,\vec{j}$ start $(-1,0,0)$

end $(+1,0,0)$

in $z=0$ plane.

along the semicircle



$$\vec{R} = (1)\cos\theta\,\vec{i} + (1)\sin\theta\,\vec{j}$$

$-\pi \le \theta \le 0$ no: lower semicircle

$\pi \le \theta \le 0$ DUMB

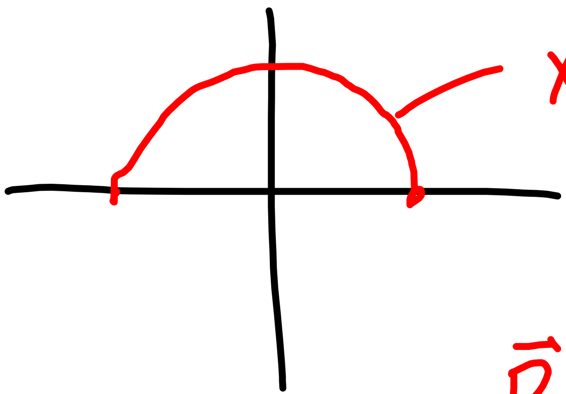
My suggestion $0 \le \theta \le \pi$, change sign of integral

Example $\vec{F} = x\,\vec{i} + x^2\,\vec{j}$ start $(-1,0,0)$

end $(+1,0,0)$

in $z=0$ plane.

along the semicircle



$x^2+y^2=1$

$y = \sqrt{1-x^2}$

$\vec{R} = x\,\vec{i} + \sqrt{1-x^2}\,\vec{j} \qquad -1 \le x \le +1$