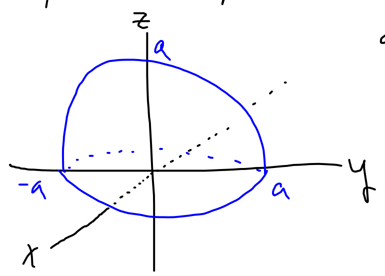
Area of a hemisphere



area $= \iint \left| \frac{\partial \vec{R}}{\partial u} \times \frac{\partial \vec{R}}{\partial v} \right| du\, dv$

$$\vec{R} = x\,\vec{i} + y\,\vec{j} + z\,\vec{k}$$

$$= a\sin\phi\cos\theta\,\vec{i} + a\sin\phi\sin\theta\,\vec{j} + a\cos\phi\,\vec{k}$$

$$= \vec{R}(\theta, \phi)$$

$$\frac{\partial \vec{R}}{\partial \theta} \times \frac{\partial \vec{R}}{\partial \phi} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -a\sin\phi\sin\theta & a\sin\phi\cos\theta & 0 \\ +a\cos\phi\cos\theta & a\cos\phi\sin\theta & -a\sin\phi \end{vmatrix}$$

$$= \vec{i}\left[-a^2\sin^2\phi\cos\theta\right] + \vec{j}\left[-a^2\sin^2\phi\sin\theta\right]$$

$$+ \vec{k}\left[-a^2\sin\phi\cos\phi\sin^2\theta - a^2\sin\phi\cos\phi\cos^2\theta\right]$$

$$\left| \frac{\partial \vec{R}}{\partial \theta} \times \frac{\partial \vec{R}}{\partial \phi} \right| = a^2 \sqrt{\underbrace{\sin^2\phi\cos^2\theta + \sin^2\phi\sin^2\theta}_{\sin^2\phi\left[\cos^2\theta + \sin^2\theta\right]} + \cos^2\phi}$$

$\cdot \sin\phi \cdot$

$$= a^2 \sin\phi$$

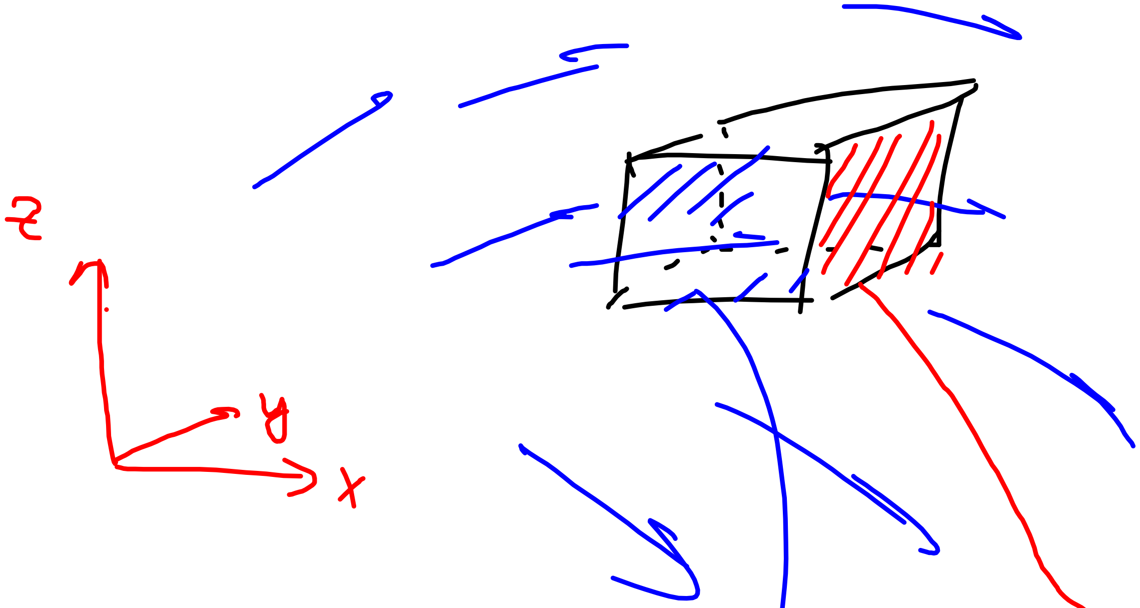
We have to evaluate

$$\int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi/2} a^2 \sin\phi \, d\theta \, d\phi$$

$$= a^2 \, 2\pi \underbrace{\Big[-\cos\phi \Big]_0^{\pi/2}}_{[-0-(-1)]}$$

$$= 2\pi a^2$$

Flux integrals.

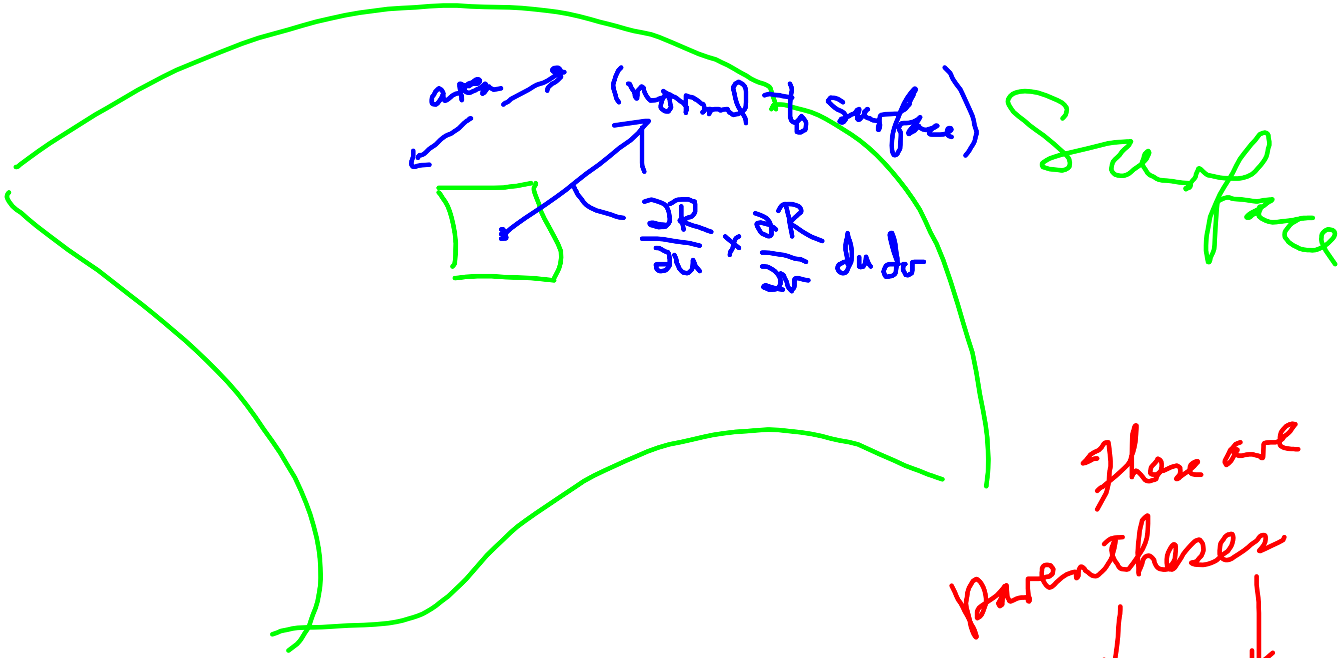


face $x = \text{constant}$

area $= \Delta y \, \Delta z$

flux $= F_1 \, \Delta y \, \Delta z$

flux $= F_2 \, \Delta x \, \Delta z \, (\pm)$



These are parentheses

Flux integral $\displaystyle\iint_{\text{surface}} F_n \cdot \text{area} = \iint \vec{F} \cdot \left(\frac{\partial \vec{R}}{\partial u} \times \frac{\partial \vec{R}}{\partial v} \right) du\, dv$

$$= \iint \underbrace{|\vec{F}| \cos\theta}_{\vec{F},\ \text{normal}} \underbrace{\left| \frac{\partial \vec{R}}{\partial u} \times \frac{\partial \vec{R}}{\partial v} \right| du\, dv}_{\text{area}}$$

Example surface: $x = u+v$ $\quad 0 \le u \le 1$

$y = u - v$

$z = u^2$ $\quad 0 \le v \le 1$

$$\vec{R} = (u+v)\vec{i} + (u-v)\vec{j} + u^2\vec{k}$$

$$\vec{F} = \vec{F}(x,y,z) = \vec{i} + xy\,\vec{j}$$

Execute $\iint \vec{F}\cdot d\vec{S}$

$d\vec{S}$ means $\frac{\partial \vec{R}}{\partial u} \times \frac{\partial \vec{R}}{\partial v}\, du\, dv$

$= \vec{n}_{\text{unit normal}} \cdot d(\text{area})$

$$= \int_u \int_v \vec{F}\cdot \frac{\partial \vec{R}}{\partial u} \times \frac{\partial \vec{R}}{\partial v}\, du\, dv$$

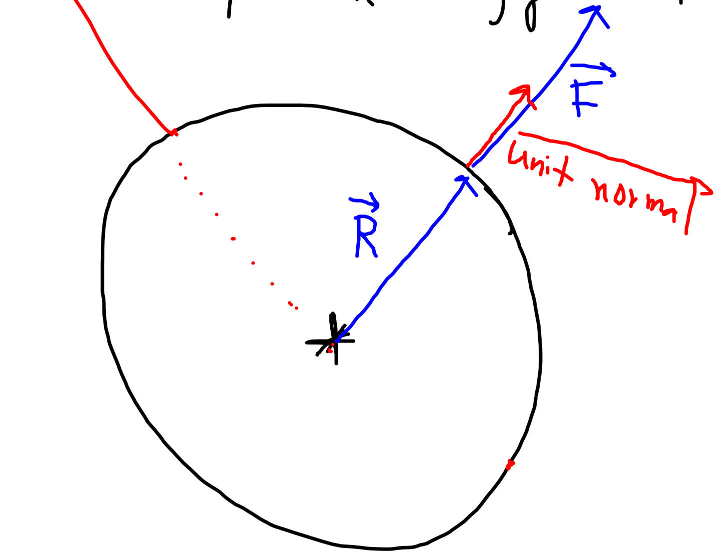
$dS = d(\text{area}) = |d\vec{S}|$

$$= \iint_{u\ v} \begin{vmatrix} F_1 & F_2 & F_3 \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix} du\, dv = \int_{u=0}^{1} \int_{v=0}^{1} \begin{vmatrix} 1 & \overset{(u+v)(u-v)}{\cancel{xy}} & 0 \\ 1 & 1 & 2u \\ 1 & -1 & 0 \end{vmatrix} du\, dv$$

Example 4.21

surface: sphere of radius 2

$$\vec F = x\vec i + y\vec j + z\vec k = \vec R$$



$$\iint \vec F \cdot d\vec S$$

$$= \iint \vec F \cdot \vec n \, d\text{area}$$

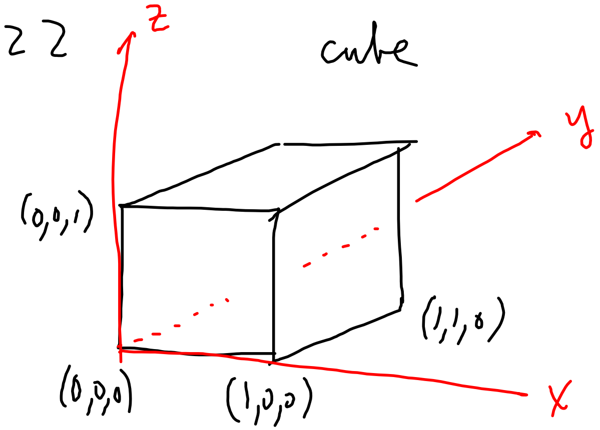
$$= \iint 2 \, d(\text{area}) = 2 \times \text{total area}$$

$$= 2 \times 4\pi \cdot 2^2$$

$$\vec F \cdot \vec n = |\vec F||\vec n| \cos\theta$$

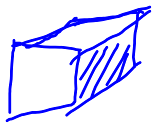
$$= 2 \cdot 1 \cdot 1$$

Example 4.22 cube



$$\vec{F} = \vec{R} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$\iint \vec{F} \cdot d\vec{S}$$

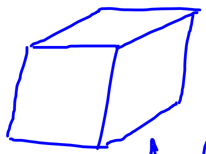


$\Rightarrow$ unit normal is $\vec{i}$

$\vec{F} \cdot \vec{n}_{unit} = x = 1 = \text{constant}$

area = 1×1

flux = $1 \cdot 1 \cdot 1 = 1$



bottom flux $= \cancel{\iint y\vec{j} \cdot (-\vec{j})\, d\,\text{area}}$

$= \cancel{\iint 0 \cdot dx\, dy} = 0$

normal $= -\vec{k}$

$\vec{F} \cdot \vec{n} = (z\vec{k}) \cdot (-\vec{k}) = -z = 0$