Dot (scalar) (inner) product :

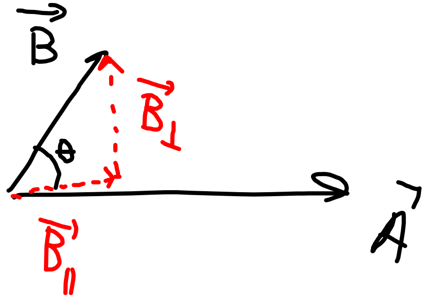
$$\vec{A} \cdot \vec{B} = |\vec{A}|\,|\vec{B}| \cos\theta \qquad \underline{\text{geom}}$$

$$= A_1 B_1 + A_2 B_2 + A_3 B_3 \qquad \underline{\text{analyt.}}$$

Cross (vector) product

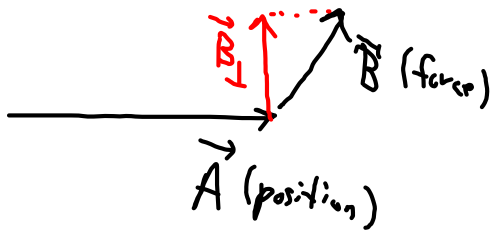
$$\vec{A} \times \vec{B}$$

geom.



$$\vec{A} \times \vec{B} = |\vec{B}_\perp|\,|\vec{A}|\,\vec{n} = |\vec{B}|\,|\vec{A}| \sin\theta \,\vec{n}$$

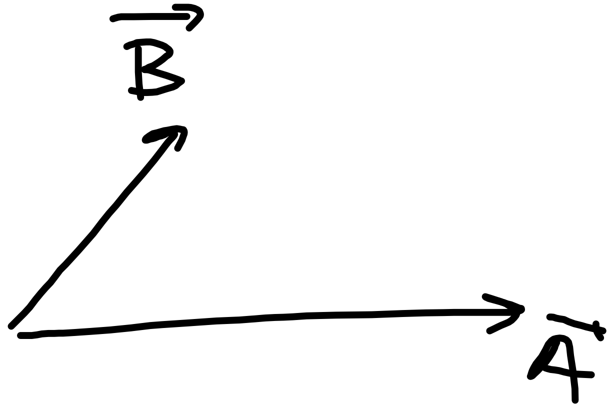
Torque



direction of $\vec{A} \times \vec{B}$ is out of the paper.

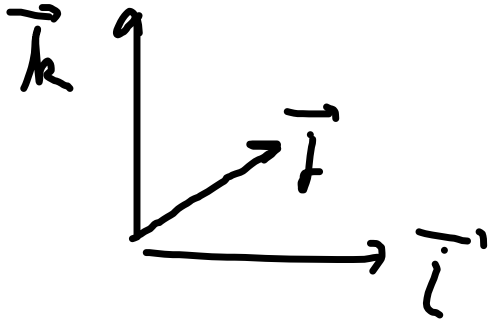
$\vec{n}$ is a unit vector $\perp$ to both $\vec{A}$ + $\vec{B}$, with $\vec{A}, \vec{B}, \vec{n}$ right-handed

Remark: a unit vector is used to designate direction.



$\vec{A} \times \vec{B}$ out of board.

$\vec{B} \times \vec{A}$ into the board.



$$\vec{i} \times \vec{j} = \vec{k}$$

$$\vec{j} \times \vec{i} = -\vec{k} \quad (= -\vec{i} \times \vec{j})$$

$$\vec{j} \times \vec{k} = \vec{i}$$

$$\vec{i} \times \vec{k} = -\vec{j}$$

$$\vec{j} \times \vec{j} = \vec{0}$$

The algebra of cross product math is straight forward, as long as you ~~keep~~ the order respect.

$$\vec{A} = A_1\vec{i} + A_2\vec{j} + A_3\vec{k}$$

$$\vec{B} = B_1\vec{i} + B_2\vec{j} + B_3\vec{k}$$

$$\vec{A}\times\vec{B} = (A_1\vec{i} + A_2\vec{j} + A_3\vec{k})\times(B_1\vec{i} + B_2\vec{j} + B_3\vec{k})$$

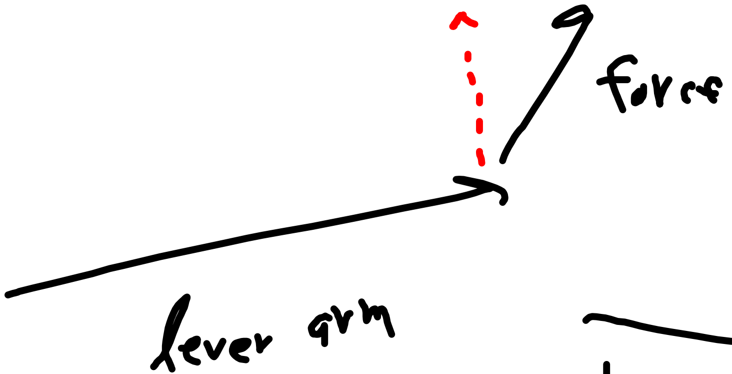
$$\vec{i}[\quad] + \vec{j}[\quad] + \vec{k}[\quad]$$

$$\underbrace{\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}}$$

$$= \vec{i}[A_2B_3 - A_3B_2] + \vec{j}[A_3B_1 - A_1B_3] + \vec{k}[A_1B_2 - A_2B_1]$$

Applications

1. Mechanics: torque = "force" × lever arm



wrong

$$\vec{\text{torque}} = \boxed{\vec{\text{force}} \times \vec{\text{lever arm}}}$$

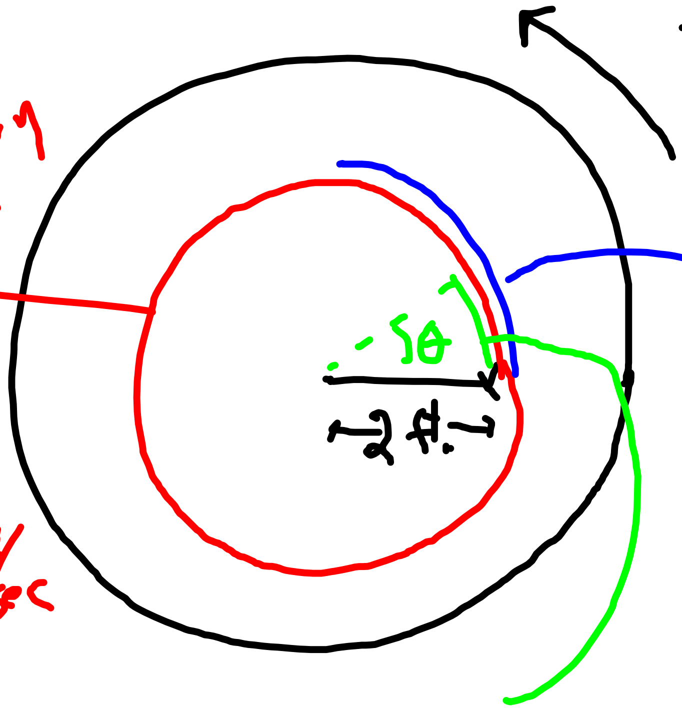
$$= \vec{\text{lever arm}} \times \vec{\text{force}}$$

360° per second

trajectory for 1 sec.

$(2 \times 2\pi \text{ ft.})$

speed $= 4\pi$ ft/sec



trajectory in $\frac{1}{4}$ sec.

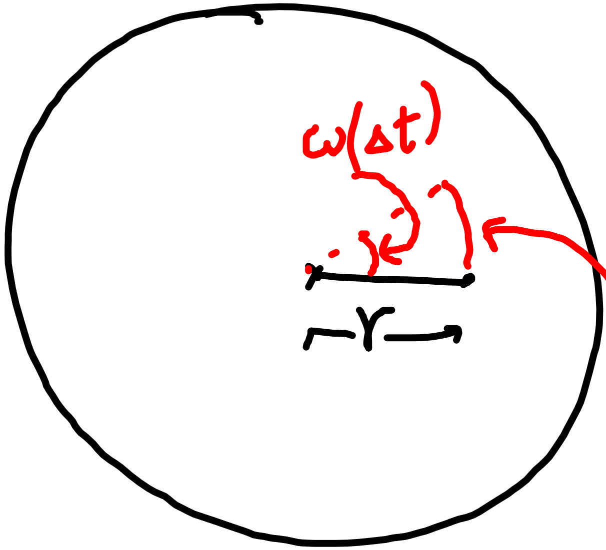
distance : $2 \times \frac{2\pi}{4}$

speed $= \pi / \tfrac{1}{4} = 4\pi$ ft/sec

trajectory over θ radians;

distance $= 2\theta$

ω rad/sec



Δt sec. time lapse

distance $= r\,\omega(\Delta t)$

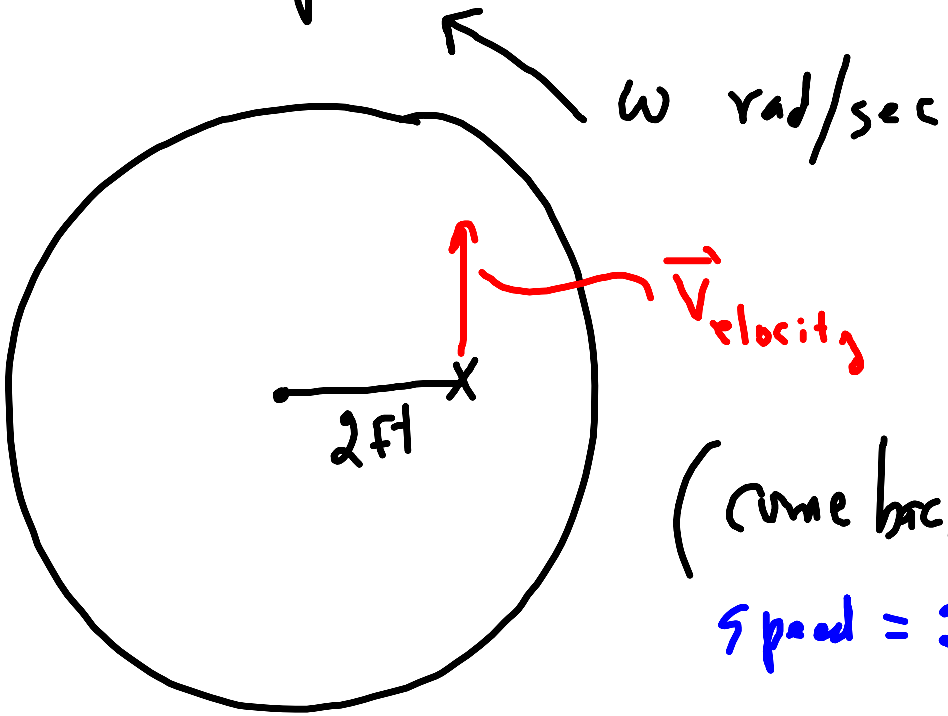
$$\text{speed} = \frac{\text{distance}}{\text{time}} = \frac{r\,\omega\,\Delta t}{\Delta t}$$

$$= \omega r$$

Go back to p. 7

2. Angular velocity.

Motion in a plane

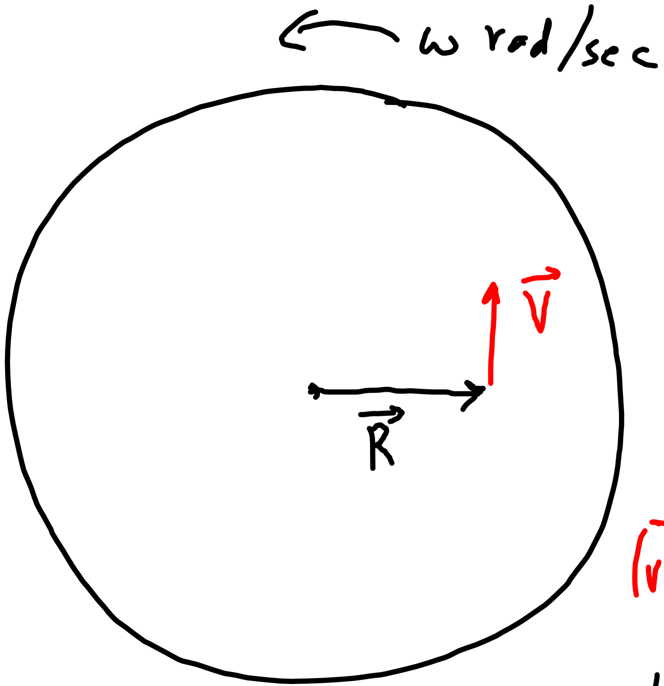


(come back to this)

speed = 2ω

$\vec{v} = 2\omega$ times a unit vector

Go to p. 10.



$$|\vec{v}| = |\vec{R}|\,\omega$$

direction of $\vec{v}$ is $\overrightarrow{(\text{vector out of paper}) \times \vec{R}}$

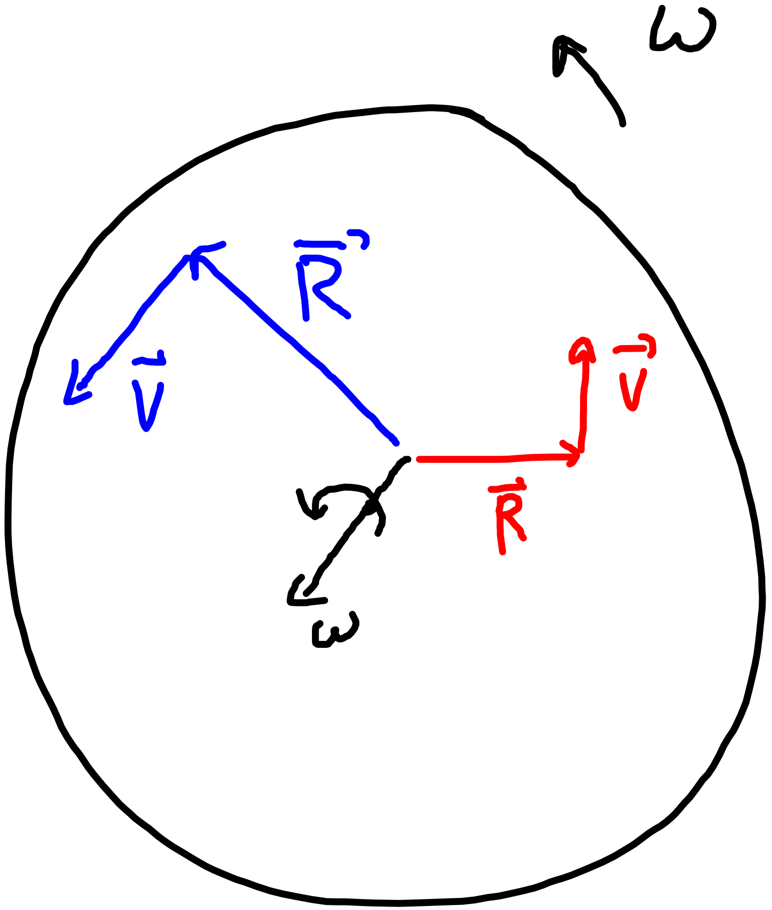
along axis of rotation

The angular velocity vector $\vec{\omega}$ is along the axis, in right-handed sense (counterclockwise rotation has axis out of page),

$$|\vec{\omega}| = \omega$$

$$\vec{v} = \vec{\omega} \times \vec{R}$$

$\vec{V} = \vec{\omega} \times \vec{R}$



$$(\vec{i} \times \vec{j}) \times \vec{i} = \vec{j}$$

$$\vec{i} \times (\vec{j} \times \vec{i}) = \vec{j}$$

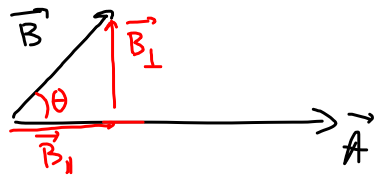
doesn't always work

$$\vec{i} \times (\vec{i} \times \vec{j}) = -\vec{j}$$

$$(\vec{i} \times \vec{i}) \times \vec{j} = \vec{0}$$

$\vec{A} \times \vec{B} \times \vec{C}$ is ~~ambig~~ ambiguous:

$$\vec{A} \times (\vec{B} \times \vec{C}) \neq (\vec{A} \times \vec{B}) \times \vec{C}$$



$$\vec{B}_\parallel = \frac{\vec{B}\cdot\vec{A}}{\vec{A}\cdot\vec{A}}\vec{A} \qquad \vec{B}_\perp = \vec{B} - \vec{B}_\parallel$$

alternative $\vec{B}_\perp$: $|\vec{B}_\perp| = |\vec{B}|\sin\theta$

direction of $\vec{B}_\perp$: picture $(\vec{B}\times\vec{A})\times\vec{A}$

opposite to $\vec{B}_\perp$

So consider $(\vec{A}\times\vec{B})\times\vec{A}$ has direction of $\vec{B}_\perp$

strategy $\vec{B}_\perp = (\text{factor})\,(\vec{A}\times\vec{B})\times\vec{A}$ end up as $|\vec{B}|\sin\theta$

magnitude of $\vec{A}\times\vec{B}$

$|\vec{A}|\,|\vec{B}|\sin\theta$

magnitude of $(\vec{A}\times\vec{B})\times\vec{A}$

mag$(\vec{A}\times\vec{B})\cdot|\vec{A}|\sin 90°$

$|A|\,|B|\sin\theta$

Answer $$\vec{B}_\perp = \frac{(\vec{A}\times\vec{B})\times\vec{A}}{|\vec{A}|^2} = \frac{(\vec{A}\times\vec{B})\times\vec{A}}{\vec{A}\cdot\vec{A}}$$

$$\vec{B}_\parallel = \frac{(\vec{A}\cdot\vec{B})\vec{A}}{\vec{A}\cdot\vec{A}}$$